Trigonometric Functions

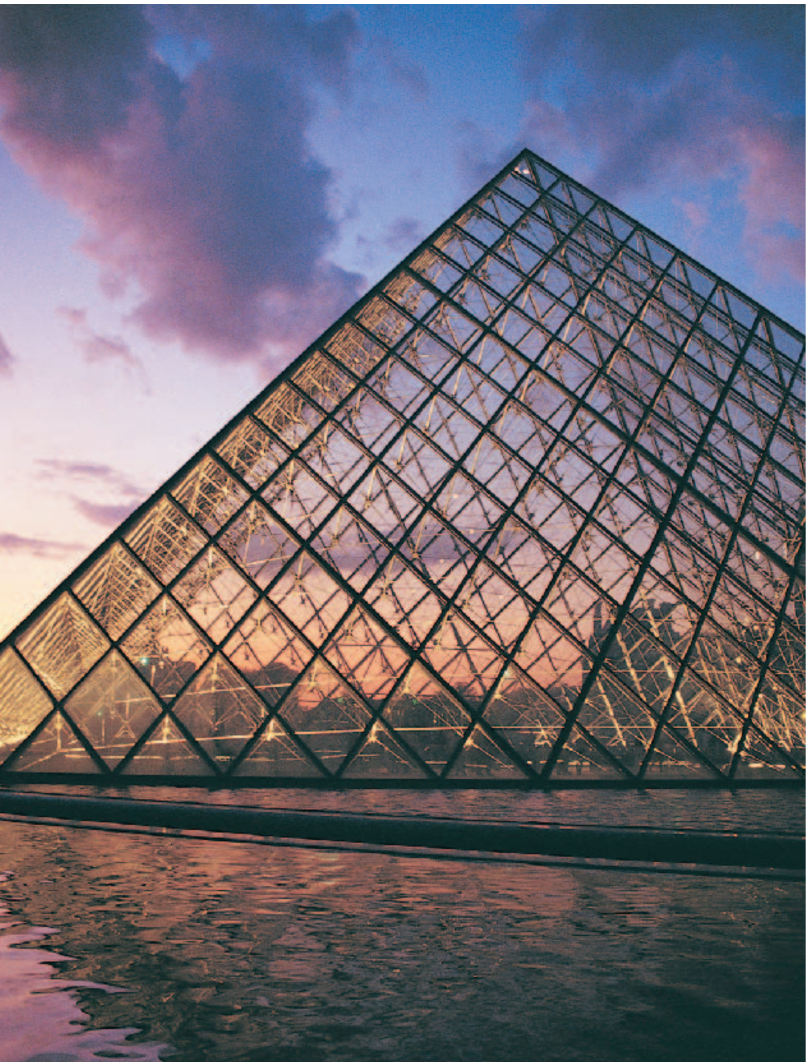
TRIGONOMETRIC functions seem to have had their origins with the Greeks' investigation of the indirect measurement of distances and angles in the "celestial sphere." (The ancient Egyptians had used some elementary geometry to build the pyramids and remeasure lands flooded by the Nile, but neither they nor the ancient Babylonians had developed the concept of angle measure.) The word *trigonometry,* based on the Greek words for "triangle measurement," was first used as the title for a text by the German mathematician Pitiscus in A.D. 1600.

Originally the trigonometric functions were restricted to angles and their applications to the indirect measurement of angles and distances. These functions gradually broke free of these restrictions, and we now have trigonometric functions of real numbers. Modern applications range over many types of problems that have little or nothing to do with angles or triangles—applications involving periodic phenomena such as sound, light, and electrical waves; business cycles; and planetary motion.

In our approach to the subject we define the trigonometric functions in terms of coordinates of points on the unit circle.

6

CHAPTER



SECTIONS

6-1 Angles and Their Measure

In Section 6-1 we introduce the concept of angle and two measures of angles, *degree* and *radian*.

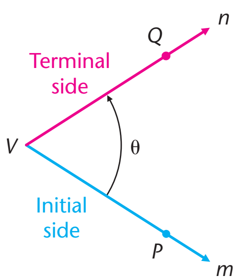


Figure 1
Angle θ or angle *PVQ* or ∠ *V*.

Angles

The study of trigonometry depends on the concept of angle. An **angle** is formed by rotating (in a plane) a ray *m*, called the **initial side** of the angle, around its endpoint until it coincides with a ray *n*, called the **terminal side** of the angle. The common endpoint *V* of *m* and *n* is called the **vertex** (Fig. 1).

A counterclockwise rotation produces a **positive angle,** and a clockwise rotation produces a **negative angle,** as shown in Figures 2(a) and 2(b). The amount of rotation in either direction is not restricted. Two different angles may have the same initial and terminal sides, as shown in Figure 2(c). Such angles are said to be **coterminal.**

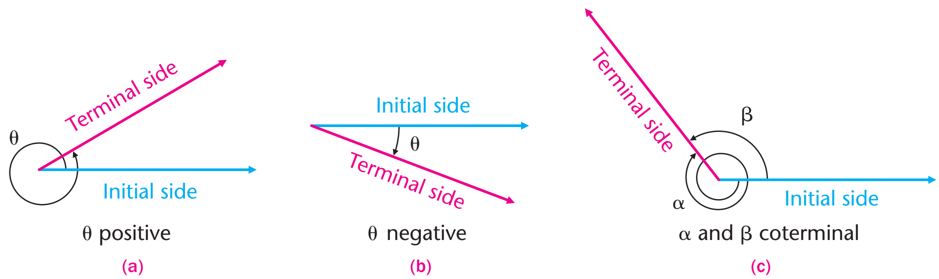


Figure 2
Angles and rotation.

An angle in a rectangular coordinate system is said to be in **standard position** if its vertex is at the origin and the initial side is along the positive *x* axis. If the terminal side of an angle in standard position lies along a coordinate axis, the angle is said to be a **quadrantal angle.** If the terminal side does not lie along a coordinate axis, then the angle is often referred to in terms of the quadrant in which the terminal side lies (Fig. 3).

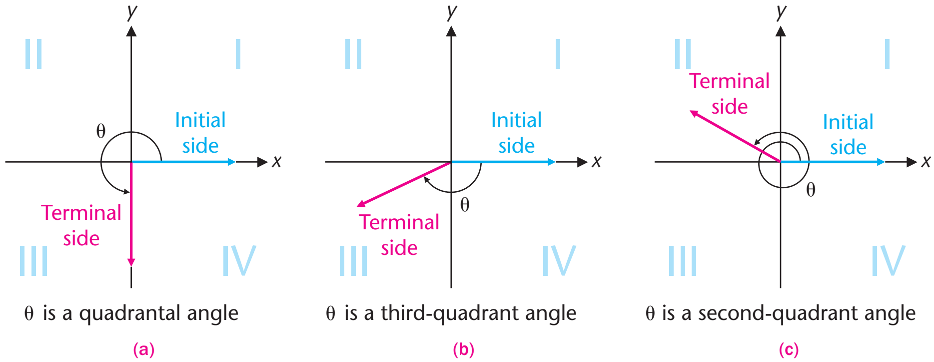


Figure 3
Angles in standard positions.

Degree and Radian Measure

Just as line segments are measured in centimeters, meters, inches, or miles, angles are measured in different units. The two most commonly used units for angle measure are *degree* and *radian*.

DEFINITION 1 Degree Measure

A positive angle formed by one complete rotation is said to have a measure of *360 degrees* (360°). A positive angle formed by $\frac{1}{360}$ of a complete rotation is said to have a measure of **1 degree** (1°). The symbol ° denotes degrees.

Definition 1 is extended to all angles, not just the positive (counterclockwise) ones, in the obvious way. So, for example, a negative angle formed by $\frac{1}{4}$ of a complete clockwise rotation has a measure of $-90°$, and an angle for which the initial and terminal sides coincide, without rotation, has a measure of 0°.

Certain angles have special names that indicate their degree measure. Figure 4 shows a **straight angle,** a **right angle,** an **acute angle,** and an **obtuse angle.**

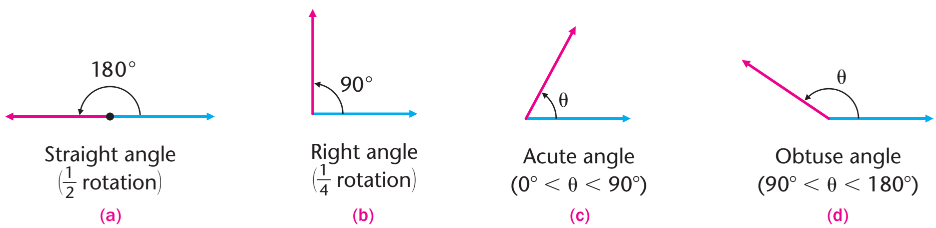


Figure 4
Types of angles.

Two positive angles are **complementary** if their sum is 90°; they are **supplementary** if their sum is 180°.

A degree can be divided further using decimal notation. For example, 42.75° represents an angle of degree measure 42 plus three-quarters of 1 degree. A degree can also be divided further using minutes and seconds just as an hour is divided into minutes and seconds. Each degree is divided into 60 equal parts called **minutes,** and each minute is divided into 60 equal parts called **seconds.** Symbolically, minutes are represented by $'$ and seconds by $''$. Thus,

$$12°23'14''$$

is a concise way of writing 12 degrees, 23 minutes, and 14 seconds.

Decimal degrees (DD) are useful in some instances and degrees–minutes–seconds (DMS) are useful in others. You should be able to go from one form to the other as demonstrated in Example 1.

CONVERSION ACCURACY

If an angle is measured to the nearest second, the converted decimal form should not go beyond three decimal places, and vice versa.

EXAMPLE 1 From DMS to DD and Back

(A) Convert 21°47′12″ to decimal degrees.

(B) Convert 105.183° to degree–minute–second form.

SOLUTIONS

(A) $21°47'12'' = \left(21 + \frac{47}{60} + \frac{12}{3{,}600}\right)^{\circ} = 21.787°$

(B) $105.183°$ $\boxed{= 105°\ (0.183 \cdot 60)'}$ *

$= 105°\ 10.98'$

$\boxed{= 105°\ 10'\ (0.98 \cdot 60)''}$

$= 105°\ 10'\ 59''$

MATCHED PROBLEM 1

(A) Convert 193°17′34″ to DD form.

(B) Convert 237.615° to DMS form.

*Throughout the book, dashed boxes—called think boxes—are used to represent steps that may be performed mentally.

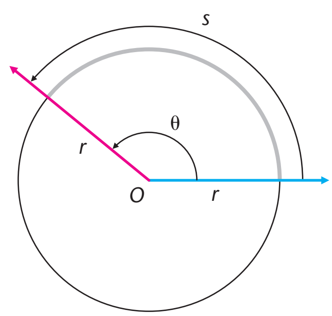
Some scientific and some graphing calculators can convert the DD and DMS forms automatically, but the process differs significantly among the various types of calculators. Check your owner's manual for your particular calculator. The conversion methods outlined in Example 1 show you the reasoning behind the process, and are sometimes easier to use than the "automatic" methods for some calculators.

Degree measure of angles is used extensively in engineering, surveying, and navigation. Another unit of angle measure, called the *radian,* is better suited for certain mathematical developments, scientific work, and engineering applications.

› DEFINITION 2 Radian Measure

If the vertex of a positive angle θ is placed at the center of a circle with radius $r > 0$, and the length of the arc opposite θ on the circumference is s, then the **radian measure** of θ is given by

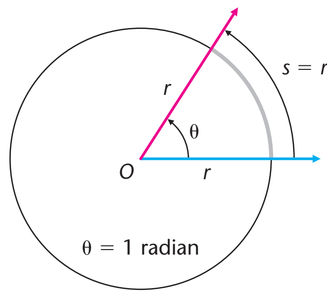
$$\theta = \frac{s}{r} \text{ radians}$$



If $s = r$, then

$$\theta = \frac{r}{r} = 1 \text{ radian}$$

Thus, **1 radian** is the measure of the central angle of a circle that intercepts an arc that has the same length as the radius of the circle. [*Note:* s and r must be measured in the same units.]



The circumference of a circle of radius r is $2\pi r$, so the radian measure of a positive angle formed by one complete rotation is

$$\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi \approx 6.283 \text{ radians}$$

Just as for degree measure, the definition is extended to apply to all angles; if θ is a negative angle, its radian measure is given by $\theta = -\frac{s}{r}$. Note that in the preceding sentence, as well as in Definition 2, the symbol θ is used in two ways: as the name of the angle and as the measure of the angle. The context indicates the meaning.

EXAMPLE 2 Computing Radian Measure

What is the radian measure of a central angle θ opposite an arc of 24 meters in a circle of radius 6 meters?

SOLUTION

$$\theta = \frac{s}{r} = \frac{24 \text{ meters}}{6 \text{ meters}} = 4 \text{ radians}$$

MATCHED PROBLEM 2

What is the radian measure of a central angle θ opposite an arc of 60 feet in a circle of radius 12 feet?

››› EXPLORE-DISCUSS 1

Discuss why the radian measure of an angle is independent of the size of the circle having the angle as a central angle.

› Converting Degrees to Radians and Vice Versa

What is the radian measure of an angle of 180°? Let θ be a central angle of 180° in a circle of radius r. Then the length s of the arc opposite θ is $\frac{1}{2}$ the circumference C of the circle. Therefore,

$$s = \frac{C}{2} = \frac{2\pi r}{2} = \pi r \qquad \text{and} \qquad \theta = \frac{s}{r} = \frac{\pi r}{r} = \pi \text{ radians}$$

Hence, 180° corresponds to π* radians. This is important to remember, because the radian measures of many special angles can be obtained from this correspondence. For example, 90° is 180°/2; therefore, 90° corresponds to $\pi/2$ radians.

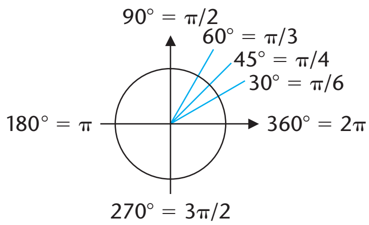
*The constant π has a long and interesting history; a few important dates are listed here:

1650 B.C.	Rhind Papyrus	$\pi \approx \frac{256}{81} = 3.16049\ldots$
240 B.C.	Archimedes	$3\frac{10}{71} < \pi < 3\frac{1}{7}$ $(3.1408\ldots < \pi < 3.1428\ldots)$
A.D. 264	Liu Hui	$\pi \approx 3.14159$
A.D. 470	Tsu Ch'ung-chih	$\pi \approx \frac{355}{113} = 3.1415929\ldots$
A.D. 1674	Leibniz	$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots)$ $\approx 3.1415926535897932384626$ (This and other series can be used to compute π to any decimal accuracy desired.)
A.D. 1761	Johann Lambert	Showed π to be irrational (π as a decimal is nonrepeating and nonterminating).

››› EXPLORE-DISCUSS 2

Write the radian measure of each of the following angles in the form $\frac{a}{b}\pi$, where a and b are positive integers and fraction $\frac{a}{b}$ is reduced to lowest terms: 15°, 30°, 45°, 60°, 75°, 90°, 105°, 120°, 135°, 150°, 165°, 180°.

Some key results from Explore-Discuss 2 are summarized in Figure 5 for easy reference. These correspondences and multiples of them will be used extensively in work that follows.



› **Figure 5**
Radian–degree correspondences.

In general, the following proportion can be used to convert degree measure to radian measure and vice versa.

› RADIAN–DEGREE CONVERSION FORMULAS

$$\frac{\theta_{\text{deg}}}{180°} = \frac{\theta_{\text{rad}}}{\pi \text{ radians}} \qquad \text{Basic proportion}$$

$$\theta_{\text{deg}} = \frac{180°}{\pi \text{ radians}}\,\theta_{\text{rad}} \qquad \text{Radians to degrees}$$

$$\theta_{\text{rad}} = \frac{\pi \text{ radians}}{180°}\,\theta_{\text{deg}} \qquad \text{Degrees to radians}$$

[*Note:* The basic proportion is usually easier to remember. Also we will omit units in calculations until the final answer. If your calculator does not have a key labeled π, use $\pi \approx 3.14159$.]

Some scientific and graphing calculators can automatically convert radian measure to degree measure, and vice versa. Check the owner's manual for your particular calculator.

EXAMPLE 3 Radian–Degree Conversions

(A) Find the radian measure, exact and to three significant digits, of an angle of 75°.

(B) Find the degree measure, exact and to four significant digits, of an angle of 5 radians.

(C) Find the radian measure to two decimal places of an angle of 41°12′.

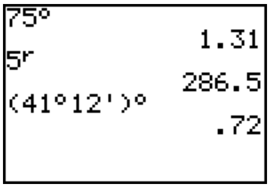
SOLUTIONS

(A) $\theta_{\text{rad}} = \dfrac{\pi \text{ radians}}{180°}\theta_{\text{deg}} = \dfrac{\pi}{180}(75) = \underbrace{\dfrac{5\pi}{12}}_{\text{Exact}} = \underbrace{1.31}_{\text{Three significant digits}}$

(B) $\theta_{\text{deg}} = \dfrac{180°}{\pi \text{ radians}}\theta_{\text{rad}} = \dfrac{180}{\pi}(5) = \underbrace{\dfrac{900}{\pi}}_{\text{Exact}} = \underbrace{286.5°}_{\text{Four significant digits}}$

(C) $41°12' = \left(41 + \dfrac{12}{60}\right)^{\circ} = 41.2°$ Change 41°12′ to DD first.

$$\theta_{\text{rad}} = \frac{\pi \text{ radians}}{180°}\theta_{\text{deg}} = \frac{\pi}{180}(41.2) = 0.72$$ To two decimal places



› **Figure 6**
Automatic conversion.

Figure 6 shows the three preceding conversions done automatically on a graphing calculator by selecting the appropriate angle mode.

MATCHED PROBLEM 3

(A) Find the radian measure, exact and to three significant digits, of an angle of 240°.

(B) Find the degree measure, exact and to three significant digits, of an angle of 1 radian.

(C) Find the radian measure to three decimal places of an angle of 125°23′.

EXAMPLE 4 Engineering

A belt connects a pulley of 2-inch radius with a pulley of 5-inch radius. If the larger pulley turns through 10 radians, through how many radians will the smaller pulley turn?

SOLUTION

First we draw a sketch (Fig. 7).

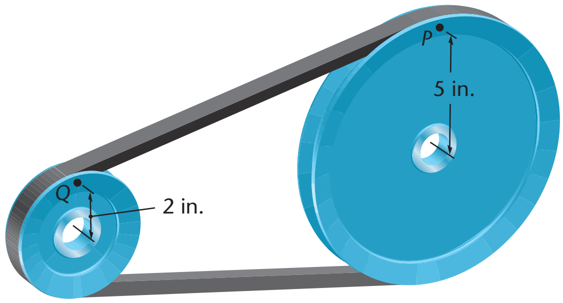


Figure 7

When the larger pulley turns through 10 radians, the point P on its circumference will travel the same distance s (arc length) that point Q on the smaller circle travels. For the larger pulley,

$$\theta = \frac{s}{r}$$
$$s = r\theta = (5)(10) = 50 \text{ inches}$$

For the smaller pulley,

$$\theta = \frac{s}{r} = \frac{50}{2} = 25 \text{ radians}$$

MATCHED PROBLEM 4

In Example 4, through how many radians will the larger pulley turn if the smaller pulley turns through 4 radians?

Linear and Angular Speed

The *average speed* v of an object that travels a distance $d = 30$ meters in time $t = 3$ seconds is given by

$$v = \frac{d}{t} = \frac{30 \text{ meters}}{3 \text{ seconds}} = 10 \text{ meters per second}$$

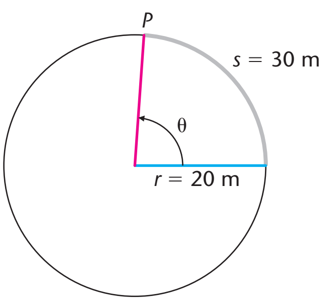


Figure 8

Suppose that a point P moves an arc length of $s = 30$ meters in $t = 3$ seconds on the circumference of a circle of radius $r = 20$ meters (Fig. 8). Then, in those 3 seconds, the point P has moved through an angle of

$$\theta = \frac{s}{r} = \frac{30}{20} = 1.5 \text{ radians}$$

We call the average speed of point P, given by

$$v = \frac{s}{t} = 10 \text{ meters per second}$$

the (average) *linear speed* to distinguish it from the (average) *angular speed* that is given by

$$\omega = \frac{\theta}{t} = \frac{1.5}{3} = 0.5 \text{ radians per second}$$

Note that $v = r\omega$ (because $s = r\theta$). These concepts are summarized in the box.

LINEAR SPEED AND ANGULAR SPEED

Suppose a point P moves through an angle θ and arc length s, in time t, on the circumference of a circle of radius r. The (average) **linear speed** of P is

$$v = \frac{s}{t}$$

and the (average) **angular speed** is

$$\omega = \frac{\theta}{t}$$

Furthermore, $v = r\omega$.

EXAMPLE 5 Wind Power

A wind turbine of rotor diameter 15 meters makes 62 revolutions per minute. Find the angular speed (in radians per second) and the linear speed (in meters per second) of the rotor tip.

SOLUTION

The radius of the rotor is $15/2 = 7.5$ meters. In 1 minute the rotor moves through an angle of $62(2\pi) = 124\pi$ radians. Therefore, the angular speed is

$$\omega = \frac{\theta}{t} = \frac{124\pi \text{ radians}}{60 \text{ seconds}} \approx 6.49 \text{ radians per second}$$

and the linear speed of the rotor tip is

$$v = r\omega = 7.5\frac{124\pi}{60} \approx 48.69 \text{ meters per second}$$

MATCHED PROBLEM 5

A wind turbine of rotor diameter 12 meters has a rotor tip speed of 34.2 meters per second. Find the angular speed of the rotor (in radians per second) and the number of revolutions per minute.

ANSWERS TO MATCHED PROBLEMS

1. (A) 193.293° (B) 237°36′54″ **2.** 5 radians
3. (A) $\frac{4\pi}{3} = 4.19$ (B) $\frac{180}{\pi} = 57.3°$ (C) 2.188
4. 1.6 radians **5.** 5.7 radians per second; 54.43 revolutions per minute

6-1 Exercises

In all problems, if angle measure is expressed by a number that is not in degrees, it is assumed to be in radians.

Find the degree measure of each of the angles in Problems 1–6, keeping in mind that an angle of one complete rotation corresponds to 360°.

1. $\frac{1}{9}$ rotation **2.** $\frac{1}{5}$ rotation **3.** $\frac{3}{4}$ rotation

4. $\frac{3}{8}$ rotation **5.** $\frac{9}{8}$ rotations **6.** $\frac{7}{6}$ rotations

Find the radian measure of a central angle θ opposite an arc s in a circle of radius r, where r and s are as given in Problems 7–10.

7. $r = 4$ centimeters, $s = 24$ centimeters

8. $r = 8$ inches, $s = 16$ inches

9. $r = 12$ feet, $s = 30$ feet

10. $r = 18$ meters, $s = 27$ meters

Find the radian measure of each angle in Problems 11–16, keeping in mind that an angle of one complete rotation corresponds to 2π radians.

11. $\frac{1}{8}$ rotation **12.** $\frac{1}{6}$ rotation **13.** $\frac{3}{4}$ rotation

14. $\frac{5}{12}$ rotation **15.** $\frac{13}{12}$ rotations **16.** $\frac{11}{8}$ rotations

Find the exact radian measure, in terms of π, of each angle in Problems 17–20.

17. 30°, 60°, 90°, 120°, 150°, 180°

18. 60°, 120°, 180°, 240°, 300°, 360°

19. −45°, −90°, −135°, −180°

20. −90°, −180°, −270°, −360°

Find the exact degree measure of each angle in Problems 21–24.

21. $\frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$ **22.** $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$

23. $-\frac{\pi}{2}, -\pi, -\frac{3\pi}{2}, -2\pi$ **24.** $-\frac{\pi}{4}, -\frac{\pi}{2}, -\frac{3\pi}{4}, -\pi$

In Problems 25–30, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

25. If two angles in standard position have the same measure, then they are coterminal.

26. If two angles in standard position are coterminal, then they have the same measure.

27. If two positive angles are complementary, then both are acute.

28. If two positive angles are supplementary, then one is obtuse and the other is acute.

29. If the terminal side of an angle in standard position lies in quadrant I, then the angle is positive.

30. If the initial and terminal sides of an angle coincide, then the measure of the angle is zero.

Convert each angle in Problems 31–34 to decimal degrees to three decimal places.

31. $5°51'33''$ **32.** $14°18'37''$

33. $354°8'29''$ **34.** $184°31'7''$

Convert each angle in Problems 35–38 to degree–minute–second form.

35. $3.042°$ **36.** $49.715°$

37. $403.223°$ **38.** $156.808°$

Find the radian measure to three decimal places for each angle in Problems 39–44.

39. $64°$ **40.** $25°$ **41.** $108.413°$

42. $203.097°$ **43.** $13°25'14''$ **44.** $56°11'52''$

Find the degree measure to two decimal places for each angle in Problems 45–50.

45. 0.93 **46.** 0.08 **47.** 1.13

48. 3.07 **49.** -2.35 **50.** -1.72

Indicate whether each angle in Problems 51–70 is a first-, second-, third-, or fourth-quadrant angle or a quadrantal angle. All angles are in standard position in a rectangular coordinate system. (A sketch may be of help in some problems.)

51. $187°$ **52.** $135°$ **53.** $-200°$

54. $-60°$ **55.** 4 **56.** 3

57. $270°$ **58.** $360°$ **59.** -1

60. -6 **61.** $\frac{5\pi}{3}$ **62.** $\frac{2\pi}{3}$

63. $-\frac{7\pi}{6}$ **64.** $-\frac{3\pi}{4}$ **65.** $-\pi$

66. $-\frac{3\pi}{2}$ **67.** $820°$ **68.** $-565°$

69. $\frac{13\pi}{4}$ **70.** $\frac{23\pi}{3}$

71. Verbally describe the meaning of a central angle in a circle with radian measure 1.

72. Verbally describe the meaning of an angle with degree measure 1.

In Problems 73–78, find all angles θ in degree measure that satisfy the given conditions.

73. $360° \le \theta \le 720°$ and θ is coterminal with $150°$

74. $360° \le \theta \le 720°$ and θ is coterminal with $240°$

75. $0° \le \theta \le 360°$ and θ is conterminal with $-80°$

76. $0° \le \theta \le 360°$ and θ is conterminal with $-310°$

77. $-900° \le \theta \le -180°$ and θ is conterminal with $210°$

78. $-900° \le \theta \le -180°$ and θ is conterminal with $135°$

In Problems 79–84, find all angles θ in radian measure that satisfy the given conditions.

79. $2\pi \le \theta \le 6\pi$ and θ is coterminal with $\pi/4$

80. $2\pi \le \theta \le 6\pi$ and θ is coterminal with $5\pi/6$

81. $0 \le \theta \le 5\pi$ and θ is coterminal with $-7\pi/6$

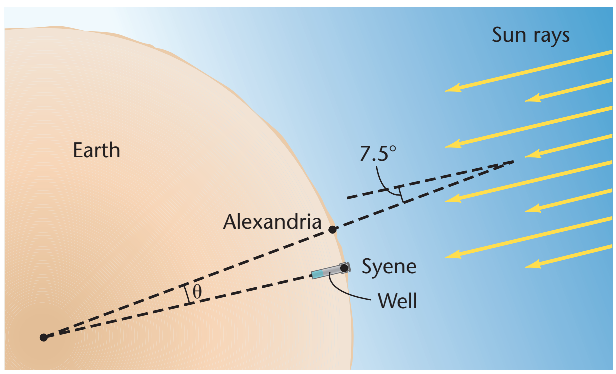
82. $0 \le \theta \le 5\pi$ and θ is coterminal with $-2\pi/3$

83. $-3\pi \le \theta \le \pi$ and θ is coterminal with $\pi/2$

84. $-3\pi \le \theta \le \pi$ and θ is coterminal with $3\pi/2$

APPLICATIONS

85. CIRCUMFERENCE OF THE EARTH The early Greeks used the proportion $s/C = \theta°/360°$, where s is an arc length on a circle, $\theta°$ is degree measure of the corresponding central angle, and C is the circumference of the circle ($C = 2\pi r$). Eratosthenes (240 B.C.), in his famous calculation of the circumference of the Earth, reasoned as follows: He knew at Syene



(now Aswan) during the summer solstice the noon sun was directly overhead and shined on the water straight down a deep well. On the same day at the same time, 5,000 stadia (approx. 500 miles) due north in Alexandria, sun rays crossed a vertical pole at an angle of 7.5° as indicated in the figure. Carry out Eratosthenes' calculation for the circumference of the Earth to the nearest thousand miles. (The current calculation for the equatorial circumference is 24,902 miles.)

86. CIRCUMFERENCE OF THE EARTH Repeat Problem 85 with the sun crossing the vertical pole in Alexandria at 7°12′.

87. CIRCUMFERENCE OF THE EARTH In Problem 85, verbally explain how θ in the figure was determined.

88. CIRCUMFERENCE OF THE EARTH Verbally explain how the radius, surface area, and volume of the Earth can be determined from the result of Problem 85.

89. ANGULAR SPEED A wheel with diameter 6 feet makes 200 revolutions per minute. Find the angular speed (in radians per second) and the linear speed (in feet per second) of a point on the rim.

90. ANGULAR SPEED A point on the rim of a wheel with diameter 6 feet has a linear speed of 100 feet per second. Find the angular speed (in radians per second) and the number of revolutions per minute.

91. RADIAN MEASURE What is the radian measure of the larger angle made by the hands of a clock at 4:30? Express the answer exactly in terms of π.

92. RADIAN MEASURE What is the radian measure of the smaller angle made by the hands of a clock at 1:30? Express the answer exactly in terms of π.

93. ENGINEERING Through how many radians does a pulley of 10-centimeter diameter turn when 10 meters of rope are pulled through it without slippage?

94. ENGINEERING Through how many radians does a pulley of 6-inch diameter turn when 4 feet of rope are pulled through it without slippage?

95. ASTRONOMY A line from the sun to the Earth sweeps out an angle of how many radians in 1 week? Assume the Earth's orbit is circular and there are 52 weeks in a year. Express the answer in terms of π and as a decimal to two decimal places.

96. ASTRONOMY A line from the center of the Earth to the equator sweeps out an angle of how many radians in 9 hours? Express the answer in terms of π and as a decimal to two decimal places.

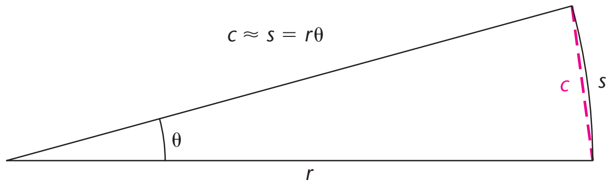
★ **97.** ENGINEERING A trail bike has a front wheel with a diameter of 40 centimeters and a back wheel of diameter 60 centimeters. Through what angle in radians does the front wheel turn if the back wheel turns through 8 radians?

★ **98.** ENGINEERING In Problem 97, through what angle in radians will the back wheel turn if the front wheel turns through 15 radians?

99. ANGULAR SPEED If the trail bike of Problem 97 travels at a speed of 10 kilometers per hour, find the angular speed (in radians per second) of each wheel.

100. ANGULAR SPEED If a car travels at a speed of 60 miles per hour, find the angular speed (in radians per second) of a tire that has a diameter of 2 feet.

The arc length on a circle is easy to compute if the corresponding central angle is given in radians and the radius of the circle is known ($s = r\theta$). If the radius of a circle is large and a central angle is small, then an arc length is often used to approximate the length of the corresponding chord as shown in the figure. If an angle is given in degree measure, converting to radian measure first may be helpful in certain problems. This information will be useful in Problems 101–104.



101. ASTRONOMY The sun is about 9.3×10^7 mi from the Earth. If the angle subtended by the diameter of the sun on the surface of the Earth is 9.3×10^{-3} rad, approximately what is the diameter of the sun to the nearest thousand miles in standard decimal notation?

102. ASTRONOMY The moon is about 381,000 kilometers from the Earth. If the angle subtended by the diameter of the moon on the surface of the Earth is 0.0092 radians, approximately what is the diameter of the moon to the nearest hundred kilometers?

103. PHOTOGRAPHY The angle of view of a 1,000-millimeter telephoto lens is 2.5°. At 750 feet, what is the width of the field of view to the nearest foot?

104. PHOTOGRAPHY The angle of view of a 300-millimeter lens is 8°. At 500 feet, what is the width of the field of view to the nearest foot?

6-2 Trigonometric Functions: A Unit Circle Approach

- The Wrapping Function
- Definitions of the Trigonometric Functions
- Graphs of the Trigonometric Functions

In Section 6-2 we introduce the six trigonometric functions in terms of the coordinates of points on the unit circle.

The Wrapping Function

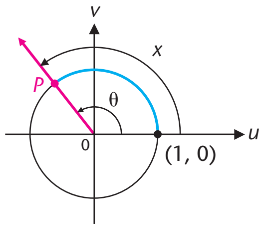


Figure 1

Consider a positive angle θ in standard position, and let P denote the point of intersection of the terminal side of θ with the unit circle $u^2 + v^2 = 1$ (Fig. 1).* Let x denote the length of the arc opposite θ on the unit circle. Because the unit circle has radius $r = 1$, the radian measure of θ is given by

$$\theta = \frac{x}{r} = \frac{x}{1} = x \text{ radians}$$

In other words, on the unit circle, the radian measure of a positive angle is equal to the length of the intercepted arc; similarly, on the unit circle, the radian measure of a negative angle is equal to the negative of the length of the intercepted arc. Because $\theta = x$, we may consider the real number x to be the name of the angle θ, when convenient. The function W that associates with each real number x the point $W(x) = P$ is called the **wrapping function.** The point P is called a **circular point.**

Consider, for example, the angle in standard position that has radian measure $\pi/2$. Its terminal side intersects the unit circle at the point (0, 1). Therefore, $W(\pi/2) = (0, 1)$. Similarly, we can find the circular point associated with any angle that is an integer multiple of $\pi/2$ (Fig. 2).

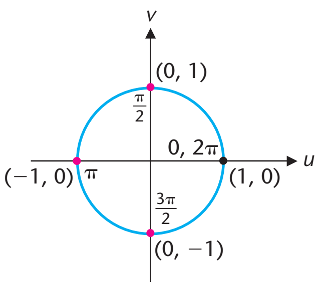


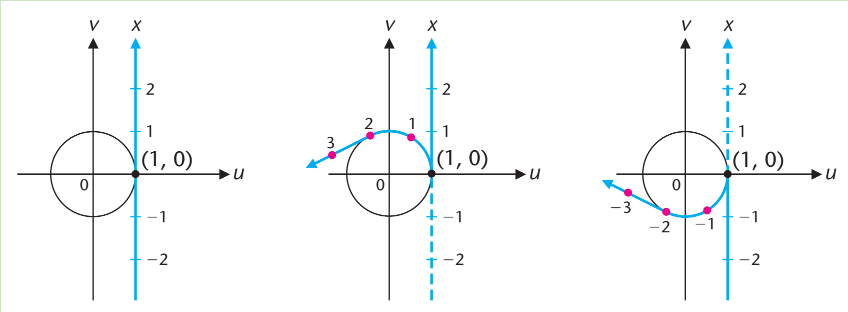
Figure 2
Circular points on the coordinate axes.

$$W(0) = (1, 0)$$
$$W\left(\frac{\pi}{2}\right) = (0, 1)$$
$$W(\pi) = (-1, 0)$$
$$W\left(\frac{3\pi}{2}\right) = (0, -1)$$
$$W(2\pi) = (1, 0)$$

*We use the variables u and v instead of x and y so that x can be used without ambiguity as an independent variable in defining the wrapping function and the trigonometric functions.

››› EXPLORE-DISCUSS 1

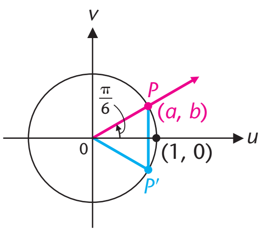
The name *wrapping function* stems from visualizing the correspondence as a wrapping of the real number line, with origin at (1, 0), around the unit circle—the positive real axis is wrapped counterclockwise, and the negative real axis is wrapped clockwise—so that each real number is paired with a unique circular point (Fig. 3).



› **Figure 3**
The wrapping function.

(A) Explain why the wrapping function is not one-to-one.

(B) In which quadrant is the circular point $W(1)$? $W(-10)$? $W(100)$?



› **Figure 4**

Given a real number x, it is difficult, in general, to find the coordinates (a, b) of the circular point $W(x)$ that is associated with x. (It is trigonometry that overcomes this difficulty.) For certain real numbers x, however, we can find the coordinates (a, b) of $W(x)$ by using simple geometric facts. For example, consider $x = \pi/6$ and let P denote the circular point $W(x) = (a, b)$ that is associated with x. Let P' be the reflection of P through the u axis (Fig. 4).

Then triangle $0PP'$ is equiangular (each angle has measure $\pi/3$ radians or 60°) and thus equilateral. Therefore $b = 1/2$. Because (a, b) lies on the unit circle, we solve for a:

$$a^2 + b^2 = 1 \qquad \text{Substitute } b = \tfrac{1}{2}.$$

$$a^2 + \left(\frac{1}{2}\right)^2 = 1 \qquad \text{Subtract } \tfrac{1}{4} \text{ from both sides.}$$

$$a^2 = \frac{3}{4} \qquad \text{Take square roots of both sides.}$$

$$a = \pm\frac{\sqrt{3}}{2} \qquad a = -\frac{\sqrt{3}}{2} \text{ must be discarded. (Why?)}$$

Thus,

$$W\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

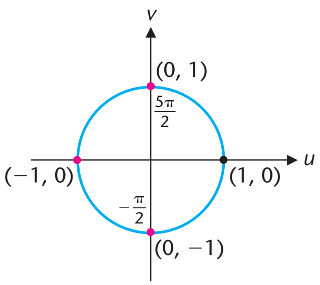
EXAMPLE 1 Coordinates of Circular Points

Find the coordinates of the following circular points:

(A) $W(-\pi/2)$ (B) $W(5\pi/2)$ (C) $W(\pi/3)$

(D) $W(7\pi/6)$ (E) $W(\pi/4)$

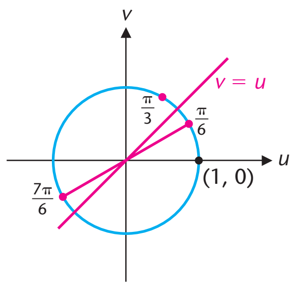
SOLUTIONS



› Figure 5

(A) Because the circumference of the unit circle is 2π, $-\pi/2$ is the radian measure of a negative angle that is $\frac{1}{4}$ of a complete clockwise rotation. Thus, $W(-\pi/2) = (0, -1)$ (Fig. 5).

(B) Starting at (1, 0) and proceeding counterclockwise, we count quarter-circle steps, $\pi/2$, $2\pi/2$, $3\pi/2$, $4\pi/2$, and end at $5\pi/2$. Thus, the circular point is on the positive vertical axis, and $W(5\pi/2) = (0, 1)$ (see Fig. 5).



› Figure 6

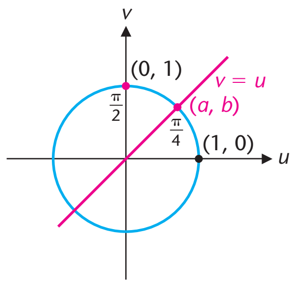
(C) The circular point $W(\pi/3)$ is the reflection of the point $W(\pi/6) = (\sqrt{3}/2, 1/2)$ through the line $u = v$. Thus, $W(\pi/3) = (1/2, \sqrt{3}/2)$ (Fig. 6).

(D) The circular point $W(7\pi/6)$ is the reflection of the point $W(\pi/6) = (\sqrt{3}/2, 1/2)$ through the origin. Thus, $W(7\pi/6) = (-\sqrt{3}/2, -1/2)$ (see Fig. 6).

(E) The circular point $W(\pi/4)$ lies on the line $u = v$, so $a = b$.

$$
\begin{aligned}
a^2 + b^2 &= 1 && \text{Substitute } b = a.\\
2a^2 &= 1 && \text{Divide both sides by 2.}\\
a^2 &= \frac{1}{2} && \text{Take square roots of both sides.}\\
a &= \frac{1}{\sqrt{2}} && a = -\frac{1}{\sqrt{2}} \text{ is impossible. (Why?)}
\end{aligned}
$$

Therefore, $W(\pi/4) = (1/\sqrt{2}, 1/\sqrt{2})$ (Fig. 7).



› Figure 7

MATCHED PROBLEM 1

Find the coordinates of the following circular points:

(A) $W(3\pi)$ (B) $W(-7\pi/2)$ (C) $W(5\pi/6)$

(D) $W(-\pi/3)$ (E) $W(5\pi/4)$

Some key results from Example 1 are summarized in Figure 8. If x is any integer multiple of $\pi/6$ or $\pi/4$, then the coordinates of $W(x)$ can be determined easily from Figure 8 by using symmetry properties.

COORDINATES OF KEY CIRCULAR POINTS

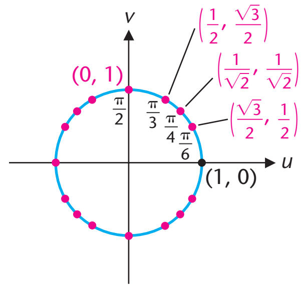


Figure 8

EXPLORE-DISCUSS 2

An effective *memory aid* for recalling the coordinates of the key circular points in Figure 8 can be created by writing the coordinates of the circular points $W(0)$, $W(\pi/6)$, $W(\pi/4)$, $W(\pi/3)$, and $W(\pi/2)$, keeping this order, in a form where each numerator is the square root of an appropriate number and each denominator is 2. For example, $W(0) = (1, 0) = (\sqrt{4}/2, \sqrt{0}/2)$. Describe the pattern that results.

Definitions of the Trigonometric Functions

We use the correspondence between real numbers and circular points to define the six **trigonometric functions: sine, cosine, tangent, cotangent, secant,** and **cosecant.** The values of these functions at a real number x are denoted by **sin *x*, cos *x*, tan *x*, cot *x*, sec *x*,** and **csc *x*,** respectively.

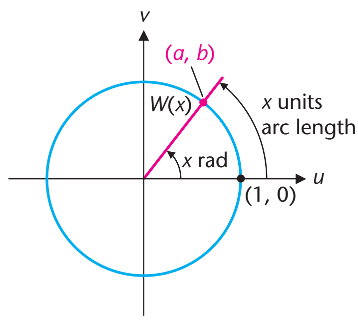
DEFINITION 1 Trigonometric Functions

Let x be a real number and let (a, b) be the coordinates of the circular point $W(x)$ that lies on the terminal side of the angle with radian measure x. Then:

$$\sin x = b \qquad \csc x = \frac{1}{b} \quad b \neq 0$$

$$\cos x = a \qquad \sec x = \frac{1}{a} \quad a \neq 0$$

$$\tan x = \frac{b}{a} \quad a \neq 0 \qquad \cot x = \frac{a}{b} \quad b \neq 0$$



The domain of both the sine and cosine functions is the set of real numbers R. The range of both the sine and cosine functions is $[-1, 1]$. This is the set of numbers assumed by b, for sine, and a, for cosine, as the circular point (a, b) moves around the unit circle. The domain of cosecant is the set of real numbers x such that b in $W(x) = (a, b)$ is not 0. Similar restrictions are made on the domains of the other three trigonometric functions. We will have more to say about the domains and ranges of all six trigonometric functions in subsequent sections.

Note from Definition 1 that $\csc x$ is the reciprocal of $\sin x$, provided that $\sin x \neq 0$. Therefore $\sin x$ is the reciprocal of $\csc x$. Similarly, $\cos x$ and $\sec x$ are reciprocals of each other, as are $\tan x$ and $\cot x$. We call these useful facts the *reciprocal identities*.

RECIPROCAL IDENTITIES

For x any real number:

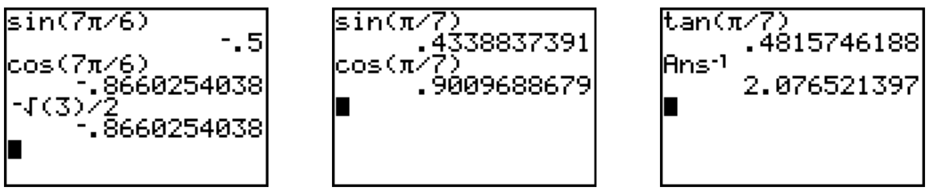
$$\csc x = \frac{1}{\sin x} \qquad \sin x \neq 0$$

$$\sec x = \frac{1}{\cos x} \qquad \cos x \neq 0$$

$$\cot x = \frac{1}{\tan x} \qquad \tan x \neq 0$$

In Example 1 we were able to give a simple geometric argument to find, for example, that the coordinates of $W(7\pi/6)$ are $(-\sqrt{3}/2, -1/2)$. Therefore, $\sin(7\pi/6) = -1/2$ and $\cos(7\pi/6) = -\sqrt{3}/2$. These exact values correspond to the approximations given by a calculator [Fig. 9(a)]. For most values of x, however, simple geometric arguments fail to give the exact coordinates of $W(x)$. But a calculator, set in radian mode, can be used to give approximations. For example, if $x = \pi/7$, then $W(\pi/7) \approx (0.901, 0.434)$ [Fig. 9(b)].

Figure 9



(a) (b) (c)

Most calculators have function keys for the sine, cosine, and tangent functions, but not for the cotangent, secant, and cosecant. Because the cotangent, secant, and cosecant are the reciprocals of the tangent, cosine, and sine, respectively, they can be evaluated easily. For example, $\cot(\pi/7) = 1/\tan(\pi/7) \approx 2.077$ [Fig. 9(c)]. Do not use the calculator function keys marked $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ for this purpose—these keys are used to evaluate the inverse trigonometric functions of Section 6-6, not reciprocals.

EXAMPLE 2

Calculator Evaluation

Evaluate to four significant digits.

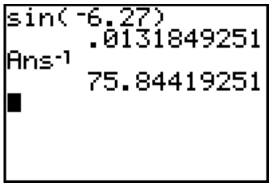
(A) tan 1.5 (B) csc (-6.27) (C) sec $(11\pi/12)$

(D) The coordinates (a, b) of $W(1)$

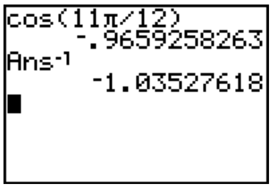
SOLUTIONS

(A) $\tan 1.5 = 14.10$

(B) $\csc(-6.27) = 1/\sin(-6.27) = 75.84$



(C) $\sec(11\pi/12) = 1/\cos(11\pi/12) = -1.035$



(D) $W(1) = (\cos 1, \sin 1) = (0.5403, 0.8415)$

MATCHED PROBLEM 2

Evaluate to four significant digits.

(A) cot (-8.25) (B) sec $(7\pi/8)$ (C) csc (4.67)

(D) The coordinates (a, b) of $W(100)$

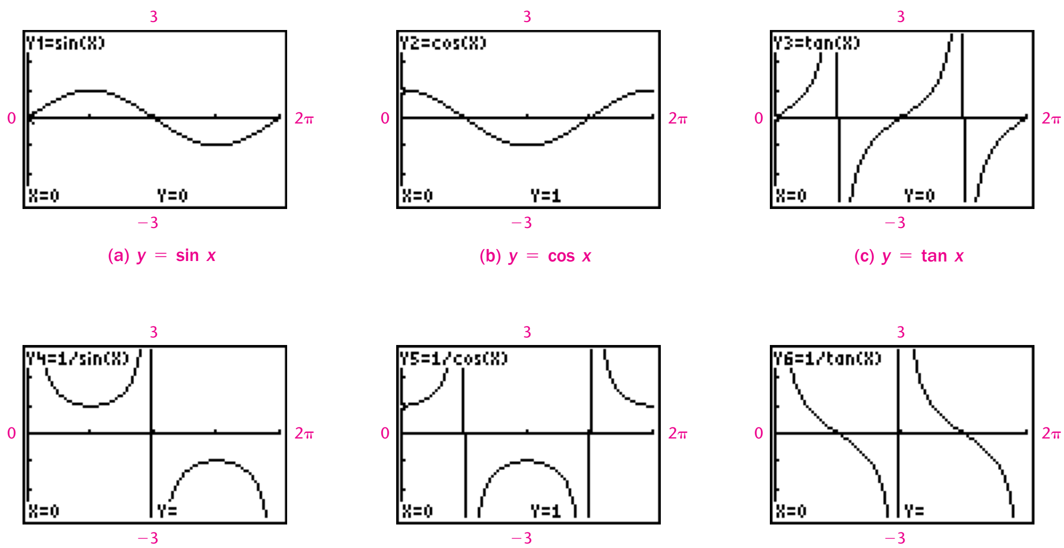
Graphs of the Trigonometric Functions

The graph of $y = \sin x$ is the set of all ordered pairs (x, y) of real numbers that satisfy the equation. Because $\sin x$, by Definition 1, is the second coordinate of the circular point $W(x)$, our knowledge of the coordinates of certain circular points (Table 1) gives the following solutions to $y = \sin x$: $(0, 0)$, $(\pi/2, 1)$, $(\pi, 0)$, and $(3\pi/2, -1)$.

Table 1

x	0	$\pi/2$	π	$3\pi/2$
$W(x)$	$(0, \mathbf{0})$	$(0, \mathbf{1})$	$(-1, \mathbf{0})$	$(0, \mathbf{-1})$
$\sin x$	$\mathbf{0}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{-1}$

As x increases from 0 to $\pi/2$, the circular point $W(x)$ moves on the circumference of the unit circle from (0, 0) to (0, 1), and so $\sin x$ [the second coordinate of $W(x)$] increases from 0 to 1. Similarly, as x increases from $\pi/2$ to π, the circular point $W(x)$ moves on the circumference of the unit circle from (0, 1) to $(-1, 0)$, and so $\sin x$ decreases from 1 to 0. These observations are in agreement with the graph of $y = \sin x$, obtained from a graphing calculator in radian mode [Fig. 10(a)].



(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \tan x$

(d) $y = \csc x$ (e) $y = \sec x$ (f) $y = \cot x$

› **Figure 10**
Graphs of the six trigonometric functions.

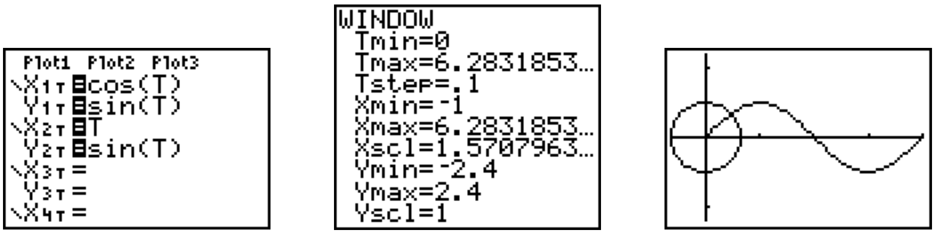
Figure 10 shows the graphs of all six trigonometric functions from $x = 0$ to $x = 2\pi$. Because the circular point $W(2\pi)$ coincides with the circular point $W(0)$, the graphs of the six trigonometric functions from $x = 2\pi$ to $x = 4\pi$ would be identical to the graphs shown in Figure 10. The functions $y = \sin x$ and $y = \cos x$ are bounded; their maximum values are 1 and their minimum values are -1. The functions $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$ are unbounded; they have vertical asymptotes at the values of x for which they are undefined. It is instructive to study and compare the graphs of reciprocal pairs, for example, $y = \cos x$ and $y = \sec x$. Note that $\sec x$ is undefined when $\cos x$ equals 0, and that because the maximum positive value of $\cos x$ is 1, the minimum positive value of $\sec x$ is 1. We will study the properties of trigonometric functions and their graphs in Section 6-4.

Technology Connections

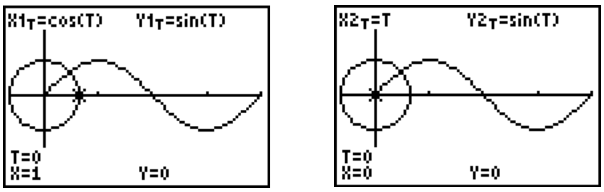
With a graphing calculator, we can illuminate the relationship between the unit circle definition of the sine function and the graph of the sine function. Set your calculator in radian and parametric modes. Make the entries indicated in Figure 11 to obtain the displayed graph (2π is entered for Tmax and Xmax, $\pi/2$ is entered for Xscl).

Use TRACE and move back and forth between the unit circle and the graph of the sine function for various values of T as T increases from 0 to 2π. Discuss what happens in each case. Figure 12 illustrates the case for $T = 0$.

Repeat the exploration with $Y_{2T} = \cos(T)$



› Figure 11



› Figure 12

EXAMPLE 3

Zeros and Turning Points

Find the zeros and turning points of $y = \cos x$ on the interval $[\pi/2, 5\pi/2]$.

SOLUTION

Recall that a *turning point* is a point on a graph that separates an increasing portion from a decreasing portion, or vice versa. As x increases from $\pi/2$ to $3\pi/2$, the first coordinate of the circular point $W(x)$ (that is, $\cos x$) decreases from 0 to a minimum

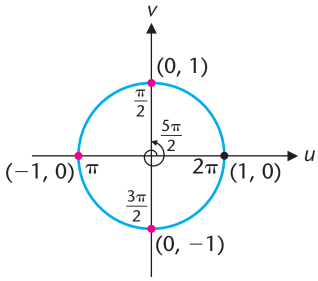


Figure 13

value of -1 (when $x = \pi$), then increases to a value of 0 (when $x = 3\pi/2$) (Fig. 13). Similarly, as x increases from $3\pi/2$ to $5\pi/2$, $\cos x$ increases from 0 to a maximum value of 1 (when $x = 2\pi$), then decreases to a value of 0. Therefore, the graph of $y = \cos x$ has turning points $(\pi, -1)$ and $(2\pi, 1)$, and zeros $\pi/2$, $3\pi/2$, and $5\pi/2$. These conclusions are confirmed by the graph of $y = \cos x$ in Figure 10(b) on page 548.

MATCHED PROBLEM 3

Find all zeros and turning points of $y = \csc x$ on the interval $(0, 4\pi)$.

ANSWERS TO MATCHED PROBLEMS

1. (A) $(-1, 0)$ (B) $(0, 1)$ (C) $(-\sqrt{3}/2, 1/2)$
(D) $(1/2, -\sqrt{3}/2)$ (E) $(-1/\sqrt{2}, -1/\sqrt{2})$
2. (A) 0.4181 (B) -1.082 (C) -1.001 (D) $(0.8623, -0.5064)$
3. Zeros: none; turning points: $(\pi/2, 1)$, $(3\pi/2, -1)$, $(5\pi/2, 1)$, $(7\pi/2, -1)$

6-2 Exercises

In Problems 1–16, find the coordinates of each circular point.

1. $W(3\pi/2)$ **2.** $W(-5\pi)$
3. $W(-6\pi)$ **4.** $W(-15\pi/2)$
5. $W(\pi/4)$ **6.** $W(\pi/3)$
7. $W(\pi/6)$ **8.** $W(-\pi/6)$
9. $W(-\pi/3)$ **10.** $W(-\pi/4)$
11. $W(2\pi/3)$ **12.** $W(11\pi/6)$
13. $W(-3\pi/4)$ **14.** $W(-7\pi/6)$
15. $W(13\pi/4)$ **16.** $W(-10\pi/3)$

In Problems 17–32, use your answers to Problems 1–16 to give the exact value of the expression (if it exists).

17. $\sin(3\pi/2)$ **18.** $\tan(-5\pi)$
19. $\cos(-6\pi)$ **20.** $\cot(-15\pi/2)$
21. $\sec(\pi/4)$ **22.** $\csc(\pi/3)$
23. $\tan(\pi/6)$ **24.** $\cos(-\pi/6)$
25. $\sin(-\pi/3)$ **26.** $\sec(-\pi/4)$
27. $\csc(2\pi/3)$ **28.** $\cot(11\pi/6)$
29. $\cos(-3\pi/4)$ **30.** $\tan(-7\pi/6)$
31. $\cot(13\pi/4)$ **32.** $\sin(-10\pi/3)$

In Problems 33–38, in which quadrants must W(x) lie so that:

33. $\cos x < 0$ **34.** $\tan x > 0$ **35.** $\sin x > 0$
36. $\sec x > 0$ **37.** $\cot x < 0$ **38.** $\csc x < 0$

Evaluate Problems 39–48 to four significant digits using a calculator set in radian mode.

39. $\cos 2.288$ **40.** $\sin 3.104$
41. $\tan(-4.644)$ **42.** $\sec(-1.555)$
43. $\csc 1.571$ **44.** $\cot 0.7854$
45. $\sin(\cos 0.3157)$ **46.** $\cos(\tan 5.183)$
47. $\cos[\csc(-1.408)]$ **48.** $\sec[\cot(-3.566)]$

Evaluate Problems 49–58 to four significant digits using a calculator. Make sure your calculator is in the correct mode (degree or radian) for each problem.

49. $\sin 25°$ **50.** $\tan 89°$

51. $\cot 12$ **52.** $\csc 13$

53. $\sin 2.137$ **54.** $\tan 4.327$

55. $\cot(-431.41°)$

56. $\sec(-247.39°)$

57. $\sin 113°27'13''$

58. $\cos 235°12'47''$

In Problems 59–64, determine whether the statement about the wrapping function W is true or false. Explain.

59. The domain of the wrapping function is the set of all points on the unit circle.

60. The domain of the wrapping function is the set of all real numbers.

61. If $W(x) = W(y)$, then $x = y$.

62. If $x = y$, then $W(x) = W(y)$.

63. If a and b are real numbers and $a^2 + b^2 = 1$, then there exists a real number x such that $W(x) = (a, b)$.

64. If a and b are real numbers and $a^2 + b^2 = 1$, then there exists a unique real number x such that $W(x) = (a, b)$.

In Problems 65–73, determine whether the statement about the trigonometric functions is true or false. Explain.

65. If x is a real number, then $\cos x$ is the reciprocal of $\sin x$.

66. If x is a real number, then $(\cot x)(\tan x) = 1$.

67. If $\sec x = \sec y$, then $x = y$.

68. If $x = y$, then $\cos x = \cos y$.

69. The functions $\sin x$ and $\csc x$ have the same domain.

70. The functions $\sin x$ and $\cos x$ have the same domain.

71. The graph of the function $\cos x$ has infinitely many turning points.

72. The graph of the function $\tan x$ has infinitely many turning points.

73. The graph of the function $\cot x$ has infinitely many zeros.

74. The graph of the function $\csc x$ has infinitely many zeros.

In Problems 75–78, find all zeros and turning points of each function on $[0, 4\pi]$.

75. $y = \sec x$ **76.** $y = \sin x$

77. $y = \tan x$ **78.** $y = \cot x$

Determine the signs of a and b for the coordinates (a, b) of each circular point indicated in Problems 79–88. First determine the quadrant in which each circular point lies. [Note: $\pi/2 \approx 1.57$, $\pi \approx 3.14$, $3\pi/2 \approx 4.71$, and $2\pi \approx 6.28$.]

79. $W(2)$ **80.** $W(1)$ **81.** $W(3)$

82. $W(4)$ **83.** $W(5)$ **84.** $W(7)$

85. $W(-2.5)$ **86.** $W(-4.5)$ **87.** $W(-6.1)$

88. $W(-1.8)$

In Problems 89–92, for each equation find all solutions for $0 \le x < 2\pi$, then write an expression that represents all solutions for the equation without any restrictions on x.

89. $W(x) = (1, 0)$ **90.** $W(x) = (-1, 0)$

91. $W(x) = (-1/\sqrt{2}, 1/\sqrt{2})$

92. $W(x) = (1/\sqrt{2}, -1/\sqrt{2})$

93. Describe in words why $W(x) = W(x + 4\pi)$ for every real number x.

94. Describe in words why $W(x) = W(x - 6\pi)$ for every real number x.

If $W(x) = (a, b)$, indicate whether the statements in Problems 95–100 are true or false. Sketching figures should help you decide.

95. $W(x + \pi) = (-a, -b)$ **96.** $W(x + \pi) = (a, b)$

97. $W(-x) = (-a, b)$ **98.** $W(-x) = (a, -b)$

99. $W(x + 2\pi) = (a, b)$

100. $W(x + 2\pi) = (-a, -b)$

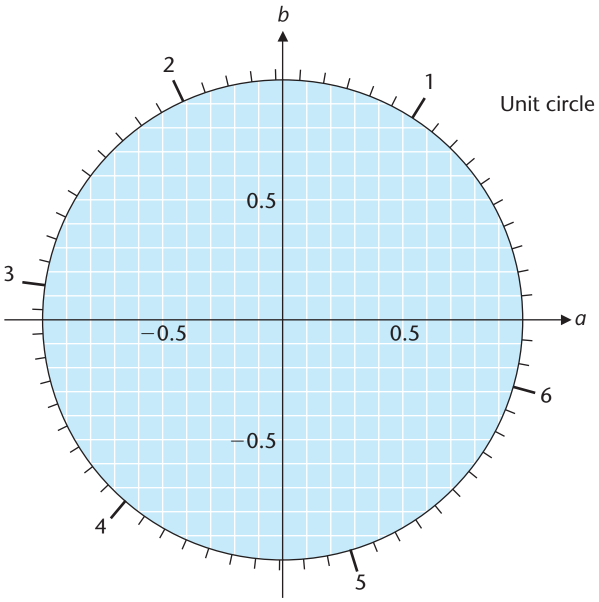
In Problems 101–104, find the value of each to one significant digit. Use only the accompanying figure on page 552, Definition 1, and a calculator as necessary for multiplication and division. Check your results by evaluating each directly on a calculator.

101. (A) $\sin 0.4$ (B) $\cos 0.4$ (C) $\tan 0.4$

102. (A) $\sin 0.8$ (B) $\cos 0.8$ (C) $\cot 0.8$

103. (A) $\sec 2.2$ (B) $\tan 5.9$ (C) $\cot 3.8$

104. (A) $\csc 2.5$ (B) $\cot 5.6$ (C) $\tan 4.3$



In Problems 105–108, in which quadrants are the statements true and why?

105. $\sin x < 0$ and $\cot x < 0$

106. $\cos x > 0$ and $\tan x < 0$

107. $\cos x < 0$ and $\sec x > 0$

108. $\sin x > 0$ and $\csc x < 0$

For which values of x, $0 \le x \le 2\pi$, is each of Problems 109–114 not defined?

109. $\cos x$ **110.** $\sin x$ **111.** $\tan x$

112. $\cot x$ **113.** $\sec x$ **114.** $\csc x$

APPLICATIONS

If an n-sided regular polygon is inscribed in a circle of radius r, then it can be shown that the area of the polygon is given by

$$A = \frac{1}{2}nr^2 \sin\frac{2\pi}{n}$$

Compute each area exactly and then to four significant digits using a calculator if the area is not an integer.

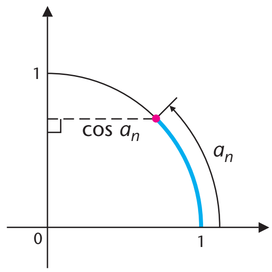
115. $n = 12$, $r = 5$ meters

116. $n = 4$, $r = 3$ inches

117. $n = 3$, $r = 4$ inches

118. $n = 8$, $r = 10$ centimeters

APPROXIMATING π *Problems 119 and 120 refer to a sequence of numbers generated as follows:*



$$\begin{aligned} &a_1 \\ &a_2 = a_1 + \cos a_1 \\ &a_3 = a_2 + \cos a_2 \\ &\vdots \\ &a_{n+1} = a_n + \cos a_n \end{aligned}$$

119. Let $a_1 = 0.5$, and compute the first five terms of the sequence to six decimal places and compare the fifth term with $\pi/2$ computed to six decimal places.

120. Repeat Problem 119, starting with $a_1 = 1$.

6-3 Solving Right Triangles*

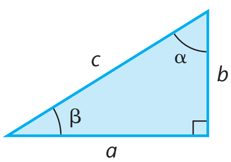


Figure 1

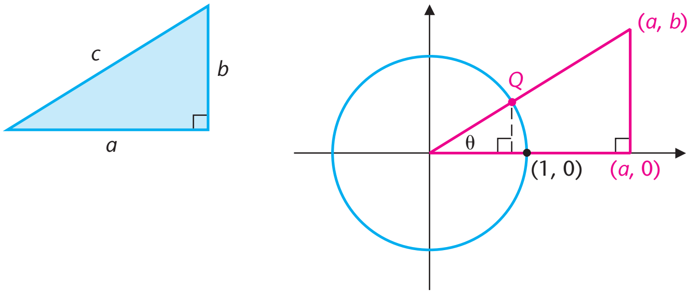
A **right triangle** is a triangle with one 90° angle (Fig. 1).

If only the angles of a right triangle are known, it is impossible to solve for the sides. (Why?) But if we are given two sides, or one acute angle and a side, then it is possible to solve for the remaining three quantities. This process is called **solving the right triangle.** We use the trigonometric functions to solve right triangles.

*This section provides a significant application of trigonometric functions to real-world problems. However, it may be postponed or omitted without loss of continuity, if desired. Some may want to cover the section just before Sections 8-1 and 8-2.

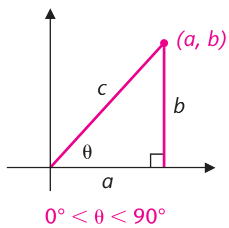
If a right triangle is located in the first quadrant as indicated by Figure 2, then, by similar triangles, the coordinates of the circular point Q are $(a/c, b/c)$.

Figure 2



Therefore, using the definition of the trigonometric functions, $\sin\theta = b/c$ and $\cos\theta = a/c$. (Calculations using such trigonometric ratios are valid if θ is measured in either degrees or radians, provided your calculator is set in the correct mode—in this section we use degree measure.) All six trigonometric ratios are displayed in the box.

TRIGONOMETRIC RATIOS



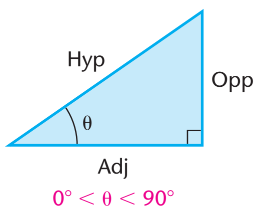
$$\sin\theta = \frac{b}{c} \qquad \csc\theta = \frac{c}{b}$$

$$\cos\theta = \frac{a}{c} \qquad \sec\theta = \frac{c}{a}$$

$$\tan\theta = \frac{b}{a} \qquad \cot\theta = \frac{a}{b}$$

Side b is often referred to as the **side opposite** angle θ, a as the **side adjacent** to angle θ, and c as the **hypotenuse.** Using these designations for an arbitrary right triangle removed from a coordinate system, we have the following:

RIGHT TRIANGLE RATIOS



$$\sin\theta = \frac{\text{Opp}}{\text{Hyp}} \qquad \csc\theta = \frac{\text{Hyp}}{\text{Opp}}$$

$$\cos\theta = \frac{\text{Adj}}{\text{Hyp}} \qquad \sec\theta = \frac{\text{Hyp}}{\text{Adj}}$$

$$\tan\theta = \frac{\text{Opp}}{\text{Adj}} \qquad \cot\theta = \frac{\text{Adj}}{\text{Opp}}$$

››› EXPLORE-DISCUSS 1

For a given value θ, $0 < \theta < 90°$, explain why the value of each of the six trigonometric functions is independent of the size of the right triangle that contains θ.

Table 1

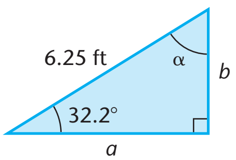
Angle to Nearest	Significant Digits for Side Measure
1°	2
10′ or 0.1°	3
1′ or 0.01°	4
10″ or 0.001°	5

The use of the trigonometric ratios for right triangles is made clear in Examples 1 through 4. Regarding computational accuracy, we use Table 1 as a guide. (The table is also printed inside the front cover of this book for easy reference.) We will use = rather than ≈ in many places, realizing the accuracy indicated in Table 1 is all that is assumed. Another word of caution: When using your calculator be sure it is set in degree mode.

EXAMPLE 1

Right Triangle Solution

Solve the right triangle with $c = 6.25$ feet and $\beta = 32.2°$.



› Figure 3

SOLUTION

First draw a figure and label the parts (Fig. 3):

SOLVE FOR α

$$\alpha = 90° - 32.2° = 57.8° \qquad \text{α and β are complementary.}$$

SOLVE FOR b

$$\sin\beta = \frac{b}{c} \qquad \text{Or use } \csc\beta = \frac{c}{b}.$$

$$\sin 32.2° = \frac{b}{6.25} \qquad \text{Multiply both sides by 6.25.}$$

$$b = 6.25 \sin 32.2° \qquad \text{Calculate.}$$

$$= 3.33 \text{ feet}$$

SOLVE FOR a

$$\cos\beta = \frac{a}{c} \qquad \text{Or use } \sec\beta = \frac{c}{a}.$$

$$\cos 32.2° = \frac{a}{6.25} \qquad \text{Multiply both sides by 6.25.}$$

$$a = 6.25 \cos 32.2° \qquad \text{Calculate.}$$

$$= 5.29 \text{ feet}$$

MATCHED PROBLEM 1

Solve the right triangle with $c = 27.3$ meters and $\alpha = 47.8°$.

In Example 2 we are confronted with a problem of the type: Find θ given

$$\sin \theta = 0.4196$$

We know how to find (or approximate) $\sin \theta$ given θ, but how do we reverse the process? How do we find θ given $\sin \theta$? First, we note that the solution to the problem can be written symbolically as either

$$\theta = \arcsin 0.4196$$

or

arcsin and $\sin^{-1}$ both represent the same thing.

$$\theta = \sin^{-1} 0.4196$$

Both expressions are read "θ is the angle whose sine is 0.4196."

››› CAUTION ›››

It is important to note that $\sin^{-1} 0.4196$ does not mean $1/(\sin 0.4196)$. The superscript $^{-1}$ is part of a function symbol, and *$\sin^{-1}$ represents the inverse sine function.* Inverse trigonometric functions are developed in detail in Section 6-6.

Fortunately, we can find θ directly using a calculator. Most calculators of the type used in this book have the function keys $\boxed{\sin^{-1}}$, $\boxed{\cos^{-1}}$, and $\boxed{\tan^{-1}}$ or their equivalents (check your manual). These function keys take us from a trigonometric ratio back to the corresponding acute angle in degree measure when the calculator is in degree mode. Thus, if $\sin \theta = 0.4196$, then we can write $\theta = \arcsin 0.4196$ or $\theta = \sin^{-1} 0.4196$. We choose the latter and proceed as follows:

$$\begin{aligned} \theta &= \sin^{-1} 0.4196 \\ &= 24.81° && \text{To the nearest hundredth degree} \\ &\text{or } 24°49' && \text{To the nearest minute} \end{aligned}$$

CHECK

$$\sin 24.81° = 0.4196$$

››› EXPLORE-DISCUSS 2

Solve each of the following for θ to the nearest hundredth of a degree using a calculator. Explain why an error message occurs in one of the problems.

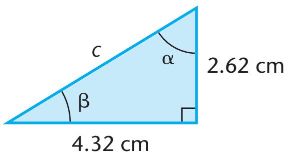
(A) $\cos\theta = 0.2044$ **(B)** $\tan\theta = 1.4138$ **(C)** $\sin\theta = 1.4138$

EXAMPLE 2 Right Triangle Solution

Solve the right triangle with $a = 4.32$ centimeters and $b = 2.62$ centimeters. Compute the angle measures to the nearest 10′.

SOLUTION

Draw a figure and label the known parts (Fig. 4):



› **Figure 4**

SOLVE FOR β

$$\tan\beta = \frac{2.62}{4.32} \qquad \text{Use } \tan^{-1} \text{ to solve for } \beta.$$

$$\beta = \tan^{-1}\frac{2.62}{4.32} \qquad \text{Calculate.}$$

$$= 31.2° \text{ or } 31°10' \qquad 0.2° = [(0.2)(60)]' = 12' \approx 10' \text{ to nearest } 10'$$

SOLVE FOR α

$$\alpha = 90° - 31°10' \quad \boxed{= 89°60' - 31°10'} \quad = 58°50'$$

SOLVE FOR *c*

$$\sin\beta = \frac{2.62}{c} \qquad \text{Or use } \csc\beta = \frac{c}{2.62}.$$

$$c = \frac{2.62}{\sin 31.2°} = 5.06 \text{ centimeters}$$

or, using the Pythagorean theorem,

$$c = \sqrt{4.32^2 + 2.62^2} = 5.05 \text{ centimeters}$$

Note the slight difference in the values obtained for c (5.05 versus 5.06). This was caused by rounding β to the nearest 10′ in the first calculation for c.

MATCHED PROBLEM 2

Solve the right triangle with $a = 1.38$ kilometers and $b = 6.73$ kilometers.

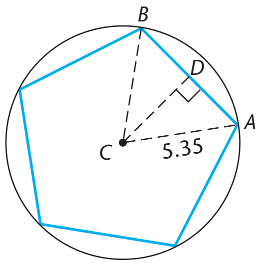
EXAMPLE 3

Geometry

If a regular pentagon (a five-sided regular polygon) is inscribed in a circle of radius 5.35 centimeters, find the length of one side of the pentagon.

SOLUTION

Sketch a figure and insert triangle ACB with C at the center (Fig. 5). Add the auxiliary line CD as indicated. We will find AD and double it to find the length of the side wanted.



› **Figure 5**

$$\text{Angle } ACB = \frac{360°}{5} = 72° \qquad \textit{Exact}$$

$$\text{Angle } ACD = \frac{72°}{2} = 36° \qquad \textit{Exact}$$

$$\sin(\text{angle } ACD) = \frac{AD}{AC}$$

$$\begin{aligned} AD &= AC \sin(\text{angle } ACD) \\ &= 5.35 \sin 36° \\ &= 3.14 \text{ centimeters} \qquad \textit{To three significant digits.} \\ AB &= 2AD = 6.28 \text{ centimeters} \end{aligned}$$

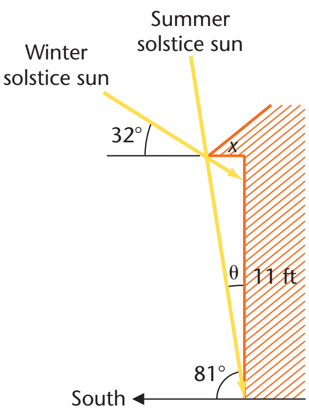
MATCHED PROBLEM 3

If a square of side 43.6 meters is inscribed in a circle, what is the radius of the circle?

EXAMPLE 4

Architecture

In designing a house an architect wishes to determine the amount of overhang of a roof so that it shades the entire south wall at noon during the summer solstice when the angle of elevation of the sun is 81° (Fig. 6). Minimally, how much overhang should be provided for this purpose?



› **Figure 6**

SOLUTION

Using Figure 6, we consider the right triangle with angle θ and sides x (the overhang) and 11 feet, and solve for x:

$$\alpha = 90° - 81° = 9°$$

$$\tan \alpha = \frac{x}{11} \qquad \textit{Multiply both sides by 11.}$$

$$x = 11 \tan 9° = 1.7 \text{ feet}$$

MATCHED PROBLEM 4

With the overhang found in Example 4, how far will the shadow of the overhang come down the wall at noon during the winter solstice when the angle of elevation of the sun is 32°?

ANSWERS TO MATCHED PROBLEMS

1. $\beta = 42.2°$, $a = 20.2$ meters, $b = 18.3$ meters
2. $\alpha = 11°40'$, $\beta = 78°20'$, $c = 6.87$ kilometers **3.** 30.8 meters **4.** 1.1 feet

6-3 Exercises

In Problems 1–6, use the figure to write the ratio of sides that corresponds to each trigonometric function. Do not look back at the definitions.

1. $\sin \theta$ **2.** $\cot \theta$ **3.** $\csc \theta$

4. $\cos \theta$ **5.** $\tan \theta$ **6.** $\sec \theta$

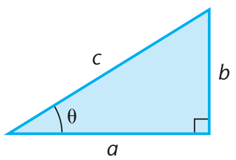


Figure for Problems 1–6

Each ratio in Problems 7–12 defines a trigonometric function of θ (refer to the figure for Problems 1–6). Indicate which function without looking back at the definitions.

7. a/c **8.** b/a **9.** c/a

10. b/c **11.** a/b **12.** c/b

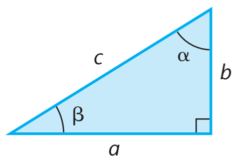
In Problems 13–18, find each acute angle θ in degree measure to two decimal places using a calculator.

13. $\cos \theta = 0.4917$ **14.** $\sin \theta = 0.0859$

15. $\theta = \tan^{-1} 8.031$ **16.** $\theta = \cos^{-1} 0.5097$

17. $\sin \theta = 0.6031$ **18.** $\tan \theta = 1.993$

In Problems 19–30, use the figure and the given information to solve each triangle.



19. $\beta = 17.8°$, $c = 3.45$ **20.** $\beta = 33.7°$, $b = 22.4$

21. $\beta = 43°20'$, $a = 123$ **22.** $\beta = 62°30'$, $c = 42.5$

23. $\alpha = 23°0'$, $a = 54.0$ **24.** $\alpha = 54°$, $c = 4.3$

25. $\alpha = 53.21°$, $b = 23.82$ **26.** $\alpha = 35.73°$, $b = 6.482$

27. $a = 6.00$, $b = 8.46$ **28.** $a = 22.0$, $b = 46.2$

29. $b = 10.0$, $c = 12.6$ **30.** $b = 50.0$, $c = 165$

In Problems 31–36, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

31. If any two angles of a right triangle are known, then it is possible to solve for the remaining angle and the three sides.

32. If any two sides of a right triangle are known, then it is possible to solve for the remaining side and the three angles.

33. If α and β are the acute angles of a right triangle, then $\sin \alpha = \sin \beta$.

34. If α and β are the acute angles of a right triangle, then $\tan \alpha = \cot \beta$.

35. If α and β are the acute angles of a right triangle, then $\sec \alpha = \cos \beta$.

36. If α and β are the acute angles of a right triangle, then $\csc \alpha = \sec \beta$.

In Problems 37–42, find the degree measure to one decimal place of the acute angle between the given line and the x axis.

37. $y = \frac{1}{2}x + 3$

38. $y = \frac{1}{3}x - \frac{1}{4}$

39. $y = 5x + 21$

40. $y = 4x - 16$

41. $y = -2x + 7$

42. $y = -3x - 1$

In Problems 43–48, find the slope to two decimal places of each line for which there is an angle of measure θ between the line and the x axis. [Hint: Note that there is an angle of measure 45° between the line y = x and the x axis, and also between the line y = −x and the x axis.]

43. $\theta = 20°$

44. $\theta = 40°$

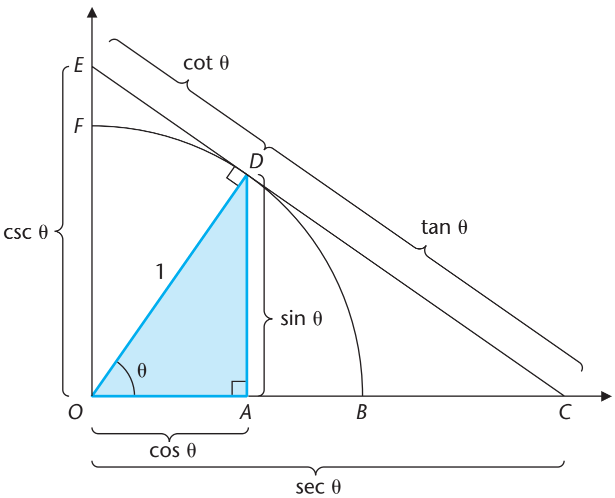
45. $\theta = 80°$

46. $\theta = 70°$

47. $\theta = \pi/30$

48. $\theta = \pi/20$

Problems 49–54 give a ***geometric interpretation of the trigonometric ratios.*** *Refer to the figure, where O is the center of a circle of radius 1, θ is the acute angle AOD, D is the intersection point of the terminal side of angle θ with the circle, and EC is tangent to the circle at D.*



49. Explain why
(A) $\cos \theta = OA$ (B) $\cot \theta = DE$ (C) $\sec \theta = OC$

50. Explain why
(A) $\sin \theta = AD$ (B) $\tan \theta = DC$ (C) $\csc \theta = OE$

51. Explain what happens to each of the following as the acute angle θ approaches 90°.
(A) $\cos \theta$ (B) $\cot \theta$ (C) $\sec \theta$

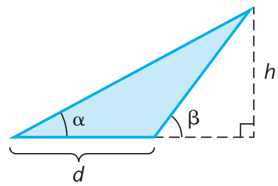
52. Explain what happens to each of the following as the acute angle θ approaches 90°.
(A) $\sin \theta$ (B) $\tan \theta$ (C) $\csc \theta$

53. Explain what happens to each of the following as the acute angle θ approaches 0°.
(A) $\sin \theta$ (B) $\tan \theta$ (C) $\csc \theta$

54. Explain what happens to each of the following as the acute angle θ approaches 0°.
(A) $\cos \theta$ (B) $\cot \theta$ (C) $\sec \theta$

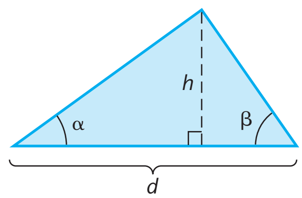
55. Show that

$$h = \frac{d}{\cot \alpha - \cot \beta}$$



56. Show that

$$h = \frac{d}{\cot \alpha + \cot \beta}$$



APPLICATIONS

57. SURVEYING Find the height of a tree (growing on level ground) if at a point 105 feet from the base of the tree the angle to its top relative to the horizontal is found to be 65.3°.

58. AIR SAFETY To measure the height of a cloud ceiling over an airport, a searchlight is directed straight upward to produce a lighted spot on the clouds. Five hundred meters away an observer reports the angle of the spot relative to the horizontal to be 32.2°. How high (to the nearest meter) are the clouds above the airport?

59. ENGINEERING If a train climbs at a constant angle of 1°23′, how many vertical feet has it climbed after going 1 mile? (1 mile = 5,280 feet)

60. AIR SAFETY If a jet airliner climbs at an angle of 15°30′ with a constant speed of 315 miles per hour, how long will it take (to the nearest minute) to reach an altitude of 8.00 miles? Assume there is no wind.

★**61.** ASTRONOMY Find the diameter of the moon (to the nearest mile) if at 239,000 miles from Earth it produces an angle of 32′ relative to an observer on Earth.

★**62.** ASTRONOMY If the sun is 93,000,000 miles from Earth and its diameter is opposite an angle of 32′ relative to an observer on Earth, what is the diameter of the sun (to two significant digits)?

★**63.** GEOMETRY If a circle of radius 4 centimeters has a chord of length 3 centimeters, find the central angle that is opposite this chord (to the nearest degree).

★**64.** GEOMETRY Find the length of one side of a nine-sided regular polygon inscribed in a circle of radius 4.06 inches.

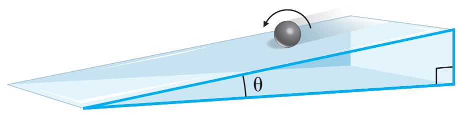
65. PHYSICS In a course in physics it is shown that the velocity v of a ball rolling down an inclined plane (neglecting air resistance and friction) is given by

$$v = gt \sin \theta$$

where g is a gravitational constant (acceleration due to gravity), t is time, and θ is the angle of inclination of the plane (see the following figure). Galileo (1564–1642) used this equation in the form

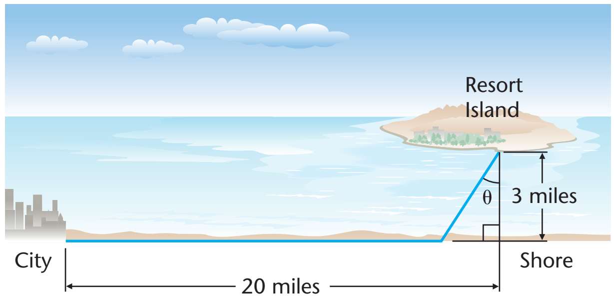
$$g = \frac{v}{t \sin \theta}$$

to estimate g after measuring v experimentally. (At that time, no timing devices existed to measure the velocity of a free-falling body, so Galileo used the inclined plane to slow the motion down.) A steel ball is rolled down a glass plane inclined at 8.0°. Approximate g to one decimal place if at the end of 3.0 seconds the ball has a measured velocity of 4.2 meters per second.



66. PHYSICS Refer to Problem 65. A steel ball is rolled down a glass plane inclined at 4.0°. Approximate g to one decimal place if at the end of 4.0 seconds the ball has a measured velocity of 9.0 feet per second.

★**67.** ENGINEERING—COST ANALYSIS A cable television company wishes to run a cable from a city to a resort island 3 miles offshore. The cable is to go along the shore, then to the island underwater, as indicated in the accompanying figure. The cost of running the cable along the shore is \$15,000 per mile and underwater, \$25,000 per mile.



(A) Referring to the figure, show that the cost in terms of θ is given by

$$C(\theta) = 75{,}000 \sec \theta - 45{,}000 \tan \theta + 300{,}000$$

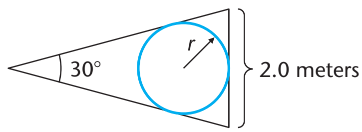
(B) Calculate a table of costs, each cost to the nearest dollar, for the following values of θ: 10°, 20°, 30°, 40°, and 50°. (Notice how the costs vary with θ. In a course in calculus, students are asked to find θ so that the cost is minimized.)

★**68.** ENGINEERING—COST ANALYSIS Refer to Problem 67. Suppose the island is 4 miles offshore and the cost of running the cable along the shore is \$20,000 per mile and underwater, \$30,000 per mile.
(A) Referring to the figure for Problem 67 with appropriate changes, show that the cost in terms of θ is given by

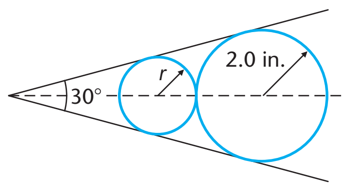
$$C(\theta) = 120{,}000 \sec \theta - 80{,}000 \tan \theta + 400{,}000$$

(B) Calculate a table of costs, each cost to the nearest dollar, for the following values of θ: 10°, 20°, 30°, 40°, and 50°.

★★**69.** GEOMETRY Find r in the accompanying figure (to two significant digits) so that the circle is tangent to all three sides of the isosceles triangle. [*Hint:* The radius of a circle is perpendicular to a tangent line at the point of tangency.]



★★**70.** GEOMETRY Find r in the accompanying figure (to two significant digits) so that the smaller circle is tangent to the larger circle and the two sides of the angle. [See the hint in Problem 69.]



6-4 Properties of Trigonometric Functions

- Basic Identities
- Sign Properties
- Periodic Functions
- Reference Triangles

In Section 6-4 we study properties of the trigonometric functions that distinguish them from the polynomial, rational, exponential, and logarithmic functions. The trigonometric functions are *periodic*, and as a consequence, have infinitely many zeros, or infinitely many turning points, or both.

Basic Identities

The definition of trigonometric functions provides several useful relationships among these functions. For convenience, we restate that definition.

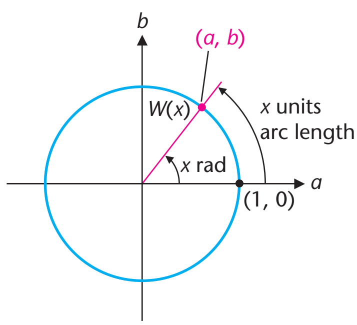
DEFINITION 1 Trigonometric Functions

Let x be a real number and let (a, b) be the coordinates of the circular point $W(x)$ that lies on the terminal side of the angle with radian measure x. Then:

$$\sin x = b \qquad\qquad \csc x = \frac{1}{b} \quad b \neq 0$$

$$\cos x = a \qquad\qquad \sec x = \frac{1}{a} \quad a \neq 0$$

$$\tan x = \frac{b}{a} \quad a \neq 0 \qquad \cot x = \frac{a}{b} \quad b \neq 0$$



Because $\sin x = b$ and $\cos x = a$, we obtain the following equations:

$$\csc x = \frac{1}{b} = \frac{1}{\sin x} \tag{1}$$

$$\sec x = \frac{1}{a} = \frac{1}{\cos x} \tag{2}$$

$$\cot x = \frac{a}{b} = \frac{1}{b/a} = \frac{1}{\tan x} \tag{3}$$

$$\tan x = \frac{b}{a} = \frac{\sin x}{\cos x} \tag{4}$$

$$\cot x = \frac{a}{b} = \frac{\cos x}{\sin x} \tag{5}$$

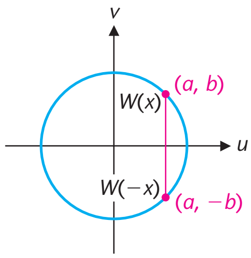


Figure 1
Symmetry property.

Because the circular points $W(x)$ and $W(-x)$ are symmetrical with respect to the horizontal axis (Fig. 1), we have the following sign properties:

$$\sin(-x) = -b = -\sin x \tag{6}$$

$$\cos(-x) = a = \cos x \tag{7}$$

$$\tan(-x) = \frac{-b}{a} = -\frac{b}{a} = -\tan x \tag{8}$$

Finally, because $(a, b) = (\cos x, \sin x)$ is on the unit circle $u^2 + v^2 = 1$, it follows that

$$(\cos x)^2 + (\sin x)^2 = 1$$

which is usually written as

$$\sin^2 x + \cos^2 x = 1 \tag{9}$$

where $\sin^2 x$ and $\cos^2 x$ are concise ways of writing $(\sin x)^2$ and $(\cos x)^2$, respectively.

››› **CAUTION** »»

$$(\sin x)^2 \neq \sin x^2$$
$$(\cos x)^2 \neq \cos x^2$$

Equations (1)–(9) are called **basic identities.** They hold true for all replacements of x by real numbers for which both sides of an equation are defined. These basic identities must be memorized along with the definitions of the six trigonometric functions, because the material is used extensively in developments that follow. Note that most of Chapter 7 is devoted to trigonometric identities.

We summarize the basic identities for convenient reference in Theorem 1.

> **THEOREM 1 Basic Trigonometric Identities**
>
> For x any real number (in all cases restricted so that both sides of an equation are defined),
>
> **Reciprocal identities**
>
> $$\csc x = \frac{1}{\sin x} \quad (1) \qquad \sec x = \frac{1}{\cos x} \quad (2) \qquad \cot x = \frac{1}{\tan x} \quad (3)$$
>
> **Quotient identities**
>
> $$\tan x = \frac{\sin x}{\cos x} \quad (4) \qquad \cot x = \frac{\cos x}{\sin x} \quad (5)$$
>
> **Identities for negatives**
>
> $$\sin(-x) = -\sin x \quad (6) \qquad \cos(-x) = \cos x \quad (7) \qquad \tan(-x) = -\tan x \quad (8)$$
>
> **Pythagorean identity**
>
> $$\sin^2 x + \cos^2 x = 1 \quad (9)$$

EXAMPLE 1 Using Basic Identities

Use the basic identities to find the values of the other five trigonometric functions given $\sin x = -\frac{1}{2}$ and $\tan x > 0$.

SOLUTION

We first note that the circular point $W(x)$ is in quadrant III, because that is the only quadrant in which $\sin x < 0$ and $\tan x > 0$. We next find $\cos x$ using identity (9):

$$\sin^2 x + \cos^2 x = 1 \qquad \text{Substitute } \sin x = -\tfrac{1}{2}.$$

$$\left(-\tfrac{1}{2}\right)^2 + \cos^2 x = 1 \qquad \text{Subtract } \tfrac{1}{4} \text{ from both sides.}$$

$$\cos^2 x = \tfrac{3}{4} \qquad \text{Take square roots of both sides.}$$

$$\cos x = -\frac{\sqrt{3}}{2} \qquad \text{Because } W(x) \text{ is in quadrant III.}$$

Now, because we have values for sin x and cos x, we can find values for the other four trigonometric functions using identities (1), (2), (4), and (5):

$$\csc x = \frac{1}{\sin x} = \frac{1}{-\frac{1}{2}} = -2 \qquad \text{Reciprocal identity (1)}$$

$$\sec x = \frac{1}{\cos x} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}} \qquad \text{Reciprocal identity (2)}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{-\frac{1}{2}}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}} \qquad \text{Quotient identity (4)}$$

$$\cot x = \frac{\cos x}{\sin x} = \frac{-\sqrt{3}/2}{-\frac{1}{2}} = \sqrt{3} \qquad \text{Quotient identity (5)}$$

[*Note:* We could also use identity (3).]

It is important to note that we were able to find the values of the other five trigonometric functions without finding x.

MATCHED PROBLEM 1

Use the basic identities to find the values of the other five trigonometric functions given $\cos x = 1/\sqrt{2}$ and $\cot x < 0$.

››› EXPLORE-DISCUSS 1

Suppose that $\sin x = -\frac{1}{2}$ and $\tan x > 0$, as in Example 1. Using basic identities and the results in Example 1, find each of the following:

(A) $\sin(-x)$ **(B)** $\sec(-x)$ **(C)** $\tan(-x)$

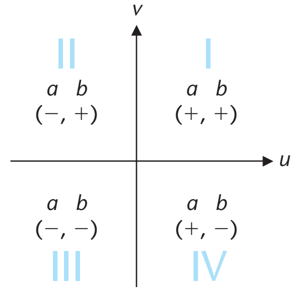
Verbally justify each step in your solution process.

› Sign Properties

As a circular point $W(x)$ moves from quadrant to quadrant, its coordinates (a, b) undergo sign changes. Hence, the trigonometric functions also undergo sign changes. It is important to know the sign of each trigonometric function in each quadrant. Table 1 shows the sign behavior for each function. It is not necessary to memorize Table 1, because the sign of each function for each quadrant is easily determined from its definition (which *should* be memorized).

Table 1 Sign Properties

Trigonometric Function	Sign in Quadrant I	II	III	IV
$\sin x = b$	+	+	−	−
$\csc x = 1/b$	+	+	−	−
$\cos x = a$	+	−	−	+
$\sec x = 1/a$	+	−	−	+
$\tan x = b/a$	+	−	+	−
$\cot x = a/b$	+	−	+	−



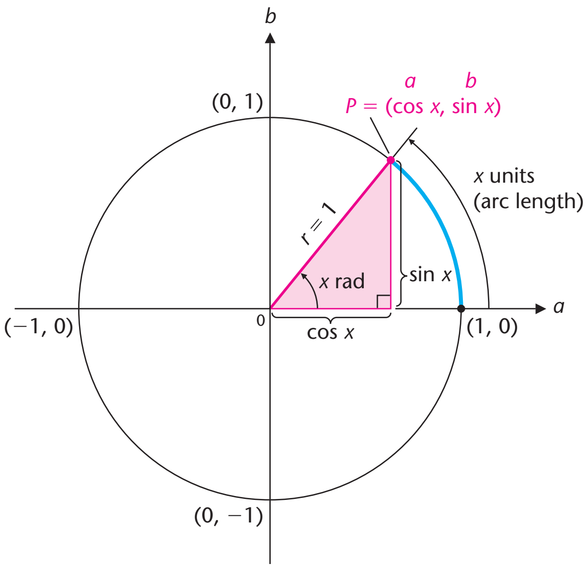
› Periodic Functions

Because the unit circle has a circumference of 2π, we find that for a given value of x (Fig. 2) we will return to the circular point $W(x) = (a, b)$ if we add any integer multiple of 2π to x. Think of a point P moving around the unit circle in either direction. Every time P covers a distance of 2π, the circumference of the circle, it is back at the point where it started. Thus, for x any real number,

$$\sin(x + 2k\pi) = \sin x \qquad k \text{ any integer}$$
$$\cos(x + 2k\pi) = \cos x \qquad k \text{ any integer}$$

Functions with this kind of repetitive behavior are called *periodic functions.* In general, we have Definition 2.

› **Figure 2**



DEFINITION 2 Periodic Functions

A function f is **periodic** if there exists a positive real number p such that

$$f(x + p) = f(x)$$

for all x in the domain of f. The smallest such positive p, if it exists, is called the **fundamental period of *f*** (or often just the **period of *f***).

Both the sine and cosine functions are periodic with period 2π. Once the graph for one period is known, the entire graph is obtained by repetition. The domain of both functions is the set of all real numbers, and the range of both is $[-1, 1]$. Because $b = 0$ at the circular points $(1, 0)$ and $(-1, 0)$, the zeros of the sine function are $k\pi$, k any integer. Because $a = 0$ at the circular points $(0, 1)$ and $(0, -1)$, the zeros of the cosine function are $\pi/2 + k\pi$, k any integer. Both the sine and cosine functions possess symmetry properties (see Section 3-3). By the basic identity $\sin(-x) = -\sin x$, the sine function is symmetric with respect to the origin, so it is an odd function. Because $\cos(-x) = \cos x$, the cosine function is symmetric with respect to the y axis, so it is an even function. Figures 3 and 4 summarize these properties and show the graphs of the sine and cosine functions, respectively.

GRAPH OF $y = \sin x$

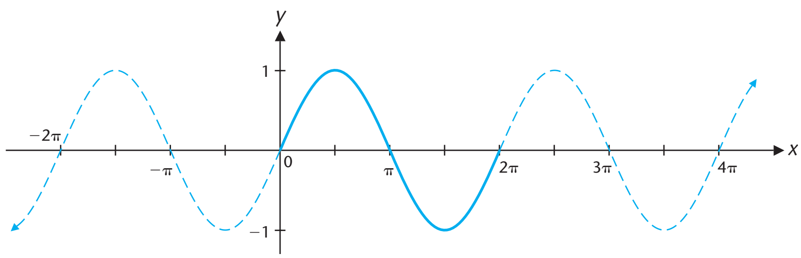


Figure 3

Period: 2π
Domain: All real numbers
Range: $[-1, 1]$

Symmetric with respect to the origin

GRAPH OF $y = \cos x$

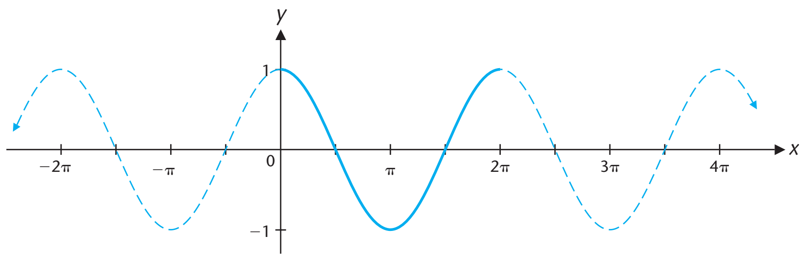


Figure 4

Period: 2π
Domain: All real numbers
Range: $[-1, 1]$

Symmetric with respect to the y axis

EXAMPLE 2 Symmetry

Determine whether the function $f(x) = \dfrac{\sin x}{x}$ is even, odd, or neither.

SOLUTION

$$
\begin{aligned}
f(-x) &= \frac{\sin(-x)}{-x} && \text{Sine function is odd.}\\
&= \frac{-\sin x}{-x}\\
&= \frac{\sin x}{x}\\
&= f(x)
\end{aligned}
$$

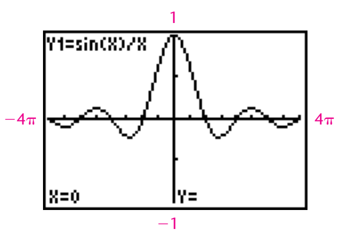


Figure 5
$y_1 = \dfrac{\sin x}{x}$

Therefore $f(x)$ is symmetric with respect to the y axis and is an even function. This fact is confirmed by the graph of $f(x)$ (Fig. 5). Note that although $f(x)$ is undefined at $x = 0$, it appears that $f(x)$ approaches 1 as x approaches 0 from either side.

MATCHED PROBLEM 2

Determine whether the function $g(x) = \dfrac{\cos x}{x}$ is even, odd, or neither.

Because the tangent function is the quotient of the sine and cosine functions, you might expect that it would also be periodic with period 2π. Surprisingly, the tangent function is periodic with period π. To see this, note that if (a, b) is the circular point associated with x, then $(-a, -b)$ is the circular point associated with $x + \pi$. Therefore,

$$\tan(x + \pi) = \frac{-b}{-a} = \frac{b}{a} = \tan x$$

The tangent function is symmetric with respect to the origin because

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x$$

Because $\tan x = \sin x/\cos x$, the zeros of the tangent function are the zeros of the sine function, namely, $k\pi$, k any integer, and the tangent function is undefined at the zeros of the cosine function, namely, $\pi/2 + k\pi$, k any integer. What does the graph of the tangent function look like near one of the values of x, say $\pi/2$, at which it is undefined? If $x < \pi/2$ but x is close to $\pi/2$, then b is close to 1 and a is positive and close to 0, so the ratio b/a is large and positive. Thus,

$$\tan x \to \infty \qquad \text{as} \qquad x \to (\pi/2)^-$$

Similarly, if $x > \pi/2$ but x is close to $\pi/2$, then b is close to 1 and a is negative and close to 0, so the ratio b/a is large in absolute value and negative. Thus,

$$\tan x \to -\infty \qquad \text{as} \qquad x \to (\pi/2)^+$$

Therefore the line $x = \pi/2$ is a vertical asymptote for the tangent function and, by periodicity, so are the vertical lines $x = \pi/2 + k\pi$, k any integer. Figure 6 summarizes these properties of the tangent function and shows its graph. The analogous properties of the cotangent function and its graph are shown in Figure 7.

GRAPH OF $y = \tan x$

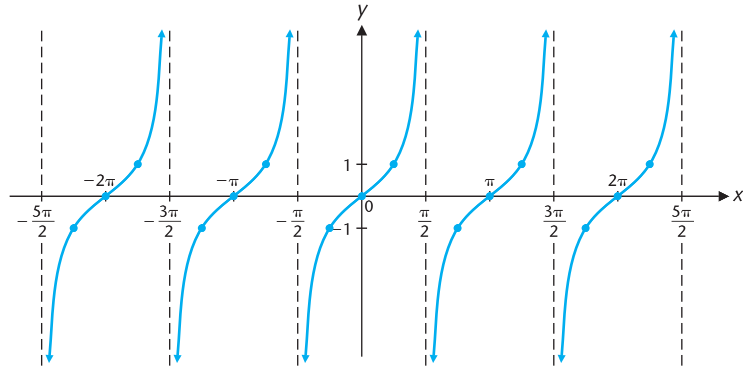


Figure 6

Period: π
Domain: All real numbers except $\pi/2 + k\pi$, k an integer
Range: All real numbers

Symmetric with respect to the origin
Increasing function between consecutive asymptotes
Discontinuous at $x = \pi/2 + k\pi$, k an integer

GRAPH OF $y = \cot x$

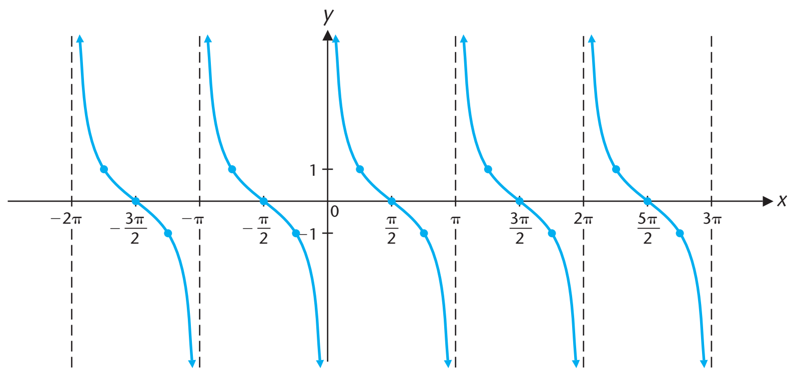


Figure 7

Period: π
Domain: All real numbers except $k\pi$, k an integer
Range: All real numbers

Symmetric with respect to the origin
Decreasing function between consecutive asymptotes
Discontinuous at $x = k\pi$, k an integer

››› EXPLORE-DISCUSS 2

(A) Discuss how the graphs of the tangent and cotangent functions are related.

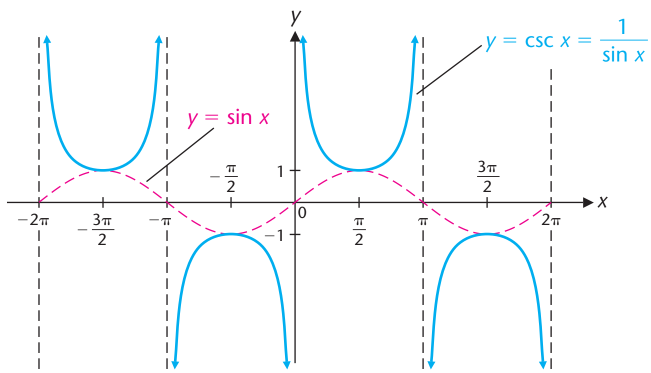
(B) How would you shift and/or reflect the tangent graph to obtain the cotangent graph?

(C) Is either the graph of $y = \tan(x - \pi/2)$ or $y = -\tan(x - \pi/2)$ the same as the graph of $y = \cot x$? Explain in terms of shifts and/or reflections.

Note that for a particular value of x, the y value on the graph of $y = \cot x$ is the reciprocal of the y value on the graph of $y = \tan x$. The vertical asymptotes of $y = \cot x$ occur at the zeros of $y = \tan x$, and vice versa.

The graphs of $y = \csc x$ and $y = \sec x$ can be obtained by taking the reciprocals of the y values of the graphs of $y = \sin x$ and $y = \cos x$, respectively. Vertical asymptotes occur at the zeros of $y = \sin x$ or $y = \cos x$. Figures 8 and 9 summarize the properties and show the graphs of $y = \csc x$ and $y = \sec x$. To emphasize the reciprocal relationships, the graphs of $y = \sin x$ and $y = \cos x$ are indicated in broken lines.

› GRAPH OF $y = \csc x$



› **Figure 8**

Period: 2π
Domain: All real numbers except $k\pi$, k an integer
Range: All real numbers y such that $y \le -1$ or $y \ge 1$

Symmetric with respect to the origin
Discontinuous at $x = k\pi$, k an integer

GRAPH OF $y = \sec x$

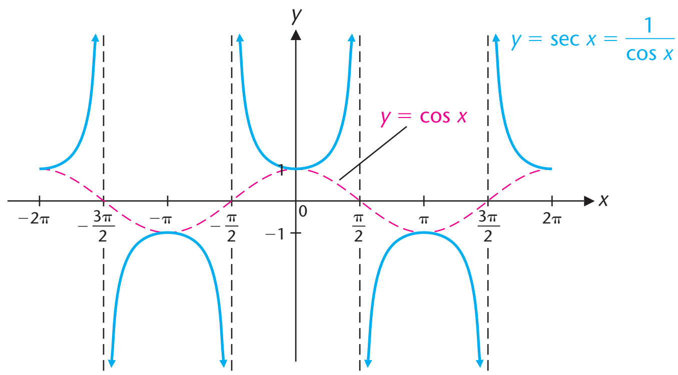


Figure 9

Period:	2π
Domain:	All real numbers except $\pi/2 + k\pi$, k an integer
Range:	All real numbers y such that $y \le -1$ or $y \ge 1$

Symmetric with respect to the y axis
Discontinuous at $x = \pi/2 + k\pi$, k an integer

Reference Triangles

Consider an angle θ in standard position. Let $P = (a, b)$ be the point of intersection of the terminal side of θ with a circle of radius $r > 0$. Then

$$a^2 + b^2 = r^2$$

so dividing both sides of the equation by r^2,

$$(a/r)^2 + (b/r)^2 = 1$$

Therefore the circular point Q on the terminal side of θ has coordinates $(a/r, b/r)$ (Fig. 10). By Definition 1,

$$\sin\theta = \frac{b}{r} \qquad \csc\theta = \frac{r}{b} \quad b \ne 0$$

$$\cos\theta = \frac{a}{r} \qquad \sec\theta = \frac{r}{a} \quad a \ne 0$$

$$\tan\theta = \frac{b}{a} \quad a \ne 0 \qquad \cot\theta = \frac{a}{b} \quad b \ne 0$$

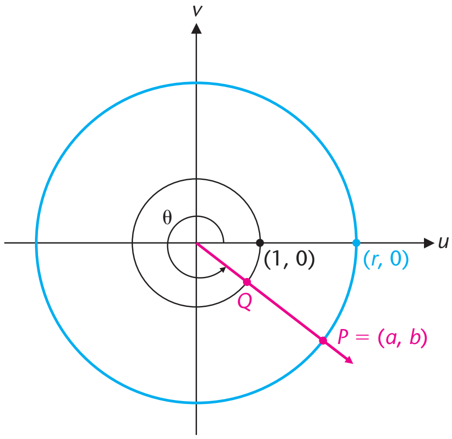
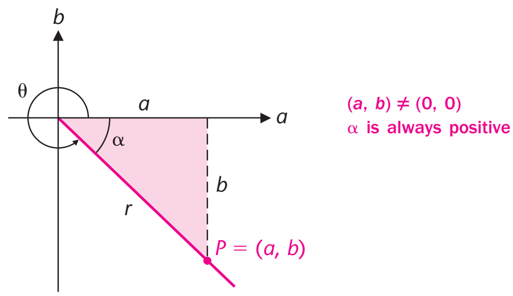


Figure 10

It is often convenient to associate a reference triangle and reference angle with θ, and to label the horizontal side, vertical side, and hypotenuse of the reference triangle with a, b, and r, respectively, to easily obtain the values of the trigonometric functions of θ.

REFERENCE TRIANGLE AND REFERENCE ANGLE

1. To form a **reference triangle** for θ, draw a perpendicular from a point $P = (a, b)$ on the terminal side of θ to the horizontal axis.
2. The **reference angle** α is the acute angle (always taken positive) between the terminal side of θ and the horizontal axis.



If Adj and Opp denote the labels a and b (possibly negative) on the horizontal and vertical sides of the reference triangle, and Hyp denotes the length r of the hypotenuse, then

$$\sin \theta = \frac{\text{Opp}}{\text{Hyp}} \qquad \csc \theta = \frac{\text{Hyp}}{\text{Opp}}$$

$$\cos \theta = \frac{\text{Adj}}{\text{Hyp}} \qquad \sec \theta = \frac{\text{Hyp}}{\text{Adj}}$$

$$\tan \theta = \frac{\text{Opp}}{\text{Adj}} \qquad \cot \theta = \frac{\text{Adj}}{\text{Opp}}$$

EXAMPLE 3 Values of the Trigonometric Functions

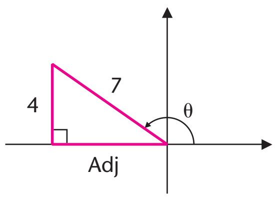
If $\sin\theta = 4/7$ and $\cos\theta < 0$, find the values of each of the other five trigonometric functions of θ.

SOLUTION

Because the sine of θ is positive and the cosine is negative, the angle θ is in quadrant II. We sketch a reference triangle (Fig. 11) and use the Pythagorean theorem to calculate the length of the horizontal side:

$$\sqrt{7^2 - 4^2} = \sqrt{33}$$

Therefore Adj $= -\sqrt{33}$, Opp $= 4$, Hyp $= 7$. The values of the other five trigonometric functions are:



› **Figure 11**

$$\cos\theta = \frac{-\sqrt{33}}{7}$$

$$\tan\theta = \frac{-4}{\sqrt{33}}$$

$$\csc\theta = \frac{7}{4}$$

$$\sec\theta = \frac{-7}{\sqrt{33}}$$

$$\cot\theta = \frac{-\sqrt{33}}{4}$$

MATCHED PROBLEM 3

If $\tan\theta = 10$ and $\sin\theta < 0$, find the values of each of the other five trigonometric functions of θ.

ANSWERS TO MATCHED PROBLEMS

1. $\sin x = -1/\sqrt{2}$, $\tan x = -1$, $\csc x = -\sqrt{2}$, $\sec x = \sqrt{2}$, $\cot x = -1$
2. Odd
3. $\sin\theta = -10/\sqrt{101}$, $\cos\theta = -1/\sqrt{101}$, $\csc\theta = -\sqrt{101}/10$, $\sec\theta = -\sqrt{101}$, $\cot\theta = 1/10$

6-4 Exercises

The figure will be useful in many of the problems in this exercise.

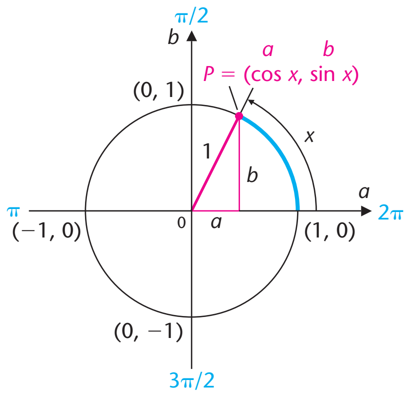


Figure for Problems 1–10.

Answer Problems 1–10 without looking back in the text or using a calculator. You can refer to the figure.

1. What are the periods of the sine, cotangent, and cosecant functions?

2. What are the periods of the cosine, tangent, and secant functions?

3. How far does the graph of each function deviate from the x axis?
(A) $y = \cos x$ (B) $y = \tan x$ (C) $y = \csc x$

4. How far does the graph of each function deviate from the x axis?
(A) $y = \sin x$ (B) $y = \cot x$ (C) $y = \sec x$

5. What are the x intercepts for the graph of each function over the interval $-2\pi \le x \le 2\pi$?
(A) $y = \sin x$ (B) $y = \cot x$ (C) $y = \csc x$

6. What are the x intercepts for the graph of each function over the interval $-2\pi \le x \le 2\pi$?
(A) $y = \cos x$ (B) $y = \tan x$ (C) $y = \sec x$

7. For what values of x, $-2\pi \le x \le 2\pi$, are the following functions not defined?
(A) $y = \cos x$ (B) $y = \tan x$ (C) $y = \csc x$

8. For what values of x, $-2\pi \le x \le 2\pi$, are the following functions not defined?
(A) $y = \sin x$ (B) $y = \cot x$ (C) $y = \sec x$

9. At what points, $-2\pi \le x \le 2\pi$, do the vertical asymptotes for the following functions cross the x axis?
(A) $y = \cos x$ (B) $y = \tan x$ (C) $y = \csc x$

10. At what points, $-2\pi \le x \le 2\pi$, do the vertical asymptotes for the following functions cross the x axis?
(A) $y = \sin x$ (B) $y = \cot x$ (C) $y = \sec x$

11. (A) Describe a shift and/or reflection that will transform the graph of $y = \csc x$ into the graph of $y = \sec x$.
(B) Is either the graph of $y = -\csc(x + \pi/2)$ or $y = -\csc(x - \pi/2)$ the same as the graph of $y = \sec x$? Explain in terms of shifts and/or reflections.

12. (A) Describe a shift and/or reflection that will transform the graph of $y = \sec x$ into the graph of $y = \csc x$.
(B) Is either the graph of $y = -\sec(x - \pi/2)$ or $y = -\sec(x + \pi/2)$ the same as the graph of $y = \csc x$? Explain in terms of shifts and/or reflections.

In Problems 13–20, determine whether each function is even, odd, or neither.

13. $y = \dfrac{\tan x}{x}$

14. $y = \dfrac{\sec x}{x}$

15. $y = \dfrac{\csc x}{x}$

16. $y = \dfrac{\cot x}{x}$

17. $y = \sin x \cos x$

18. $y = x \sin x \cos x$

19. $y = x^2 \sin x$

20. $y = x^3 \sin x$

Find the value of each of the six trigonometric functions for an angle θ that has a terminal side containing the point indicated in Problems 21–24.

21. $(6, 8)$

22. $(-3, 4)$

23. $(-1, \sqrt{3})$

24. $(\sqrt{3}, 1)$

Find the reference angle α for each angle θ in Problems 25–30.

25. $\theta = 300°$

26. $\theta = 135°$

27. $\theta = \dfrac{7\pi}{6}$

28. $\theta = \dfrac{\pi}{4}$

29. $\theta = -\dfrac{5\pi}{3}$

30. $\theta = -\dfrac{5\pi}{4}$

In Problems 31–36, find the smallest positive θ in degree and radian measure for which

31. $\cos\theta = \frac{-1}{2}$ **32.** $\sin\theta = \frac{-\sqrt{3}}{2}$

33. $\sin\theta = \frac{-1}{2}$ **34.** $\tan\theta = -\sqrt{3}$

35. $\csc\theta = \frac{-2}{\sqrt{3}}$ **36.** $\sec\theta = -\sqrt{2}$

Find the value of each of the other five trigonometric functions for an angle θ, without finding θ, given the information indicated in Problems 37–40. Sketching a reference triangle should be helpful.

37. $\sin\theta = \frac{3}{5}$ and $\cos\theta < 0$

38. $\tan\theta = -\frac{4}{3}$ and $\sin\theta < 0$

39. $\cos\theta = -\sqrt{5}/3$ and $\cot\theta > 0$

40. $\cos\theta = -\sqrt{5}/3$ and $\tan\theta > 0$

41. Which trigonometric functions are not defined when the terminal side of an angle lies along the vertical axis. Why?

42. Which trigonometric functions are not defined when the terminal side of an angle lies along the horizontal axis? Why?

43. Find exactly, all θ, $0° \le \theta < 360°$, for which $\cos\theta = -\sqrt{3}/2$.

44. Find exactly, all θ, $0° \le \theta < 360°$, for which $\cot\theta = -1/\sqrt{3}$.

45. Find exactly, all θ, $0 \le \theta < 2\pi$, for which $\tan\theta = 1$.

46. Find exactly, all θ, $0 \le \theta < 2\pi$, for which $\sec\theta = -\sqrt{2}$.

In Problems 47–56, determine whether the statement is true or false. Explain.

47. Each of the six trigonometric functions has infinitely many zeros.

48. Each of the six trigonometric functions has infinitely many turning points.

49. If a function f is periodic with period p, then $f(x) = f(x + 2p)$ for all x in the domain of f.

50. If a function f is periodic with period p, then $f(x) = f(x + p/2)$ for all x in the domain of f.

51. If the function f is not even, then it is odd.

52. The constant function with value 0 is both even and odd.

53. If f and g are each periodic with period p, then the function f/g is periodic with period p.

54. If f and g are each periodic with period p, then the function f/g is periodic.

55. If f and g are both odd, then the function fg is even.

56. If f and g are both even, then the function fg is odd.

57. Find all functions of the form $f(x) = ax + b$ that are periodic.

58. Find all functions of the form $f(x) = ax^2 + bx + c$ that are periodic.

59. Find all functions of the form $f(x) = ax + b$ that are even.

60. Find all functions of the form $f(x) = ax + b$ that are odd.

Problems 61–66 offer a preliminary investigation into the relationships of the graphs of $y = \sin x$ and $y = \cos x$ with the graphs of $y = A \sin x$, $y = A \cos x$, $y = \sin Bx$, $y = \cos Bx$, $y = \sin (x + C)$, and $y = \cos (x + C)$. This important topic is discussed in detail in Section 6-5.

61. (A) Graph $y = A \cos x$, $(-2\pi \le x \le 2\pi, -3 \le y \le 3)$, for $A = 1, 2$, and -3, all in the same viewing window.
(B) Do the x intercepts change? If so, where?
(C) How far does each graph deviate from the x axis? (Experiment with additional values of A.)
(D) Describe how the graph of $y = \cos x$ is changed by changing the values of A in $y = A \cos x$?

62. (A) Graph $y = A \sin x$, $(-2\pi \le x \le 2\pi, -3 \le y \le 3)$, for $A = 1, 3$, and -2, all in the same viewing window.
(B) Do the x intercepts change? If so, where?
(C) How far does each graph deviate from the x axis? (Experiment with additional values of A.)
(D) Describe how the graph of $y = \sin x$ is changed by changing the values of A in $y = A \sin x$?

63. (A) Graph $y = \sin Bx$ $(-\pi \le x \le \pi, -2 \le y \le 2)$, for $B = 1, 2$, and 3, all in the same viewing window.
(B) How many periods of each graph appear in this viewing rectangle? (Experiment with additional positive integer values of B.)
(C) Based on the observations in part B, how many periods of the graph of $y = \sin nx$, n a positive integer, would appear in this viewing window?

64. (A) Graph $y = \cos Bx$ $(-\pi \le x \le \pi, -2 \le y \le 2)$, for $B = 1, 2$, and 3, all in the same viewing window.
(B) How many periods of each graph appear in this viewing rectangle? (Experiment with additional positive integer values of B.)
(C) Based on the observations in part B, how many periods of the graph of $y = \cos nx$, n a positive integer, would appear in this viewing window?

65. (A) Graph $y = \cos (x + C)$, $-2\pi \le x \le 2\pi$, $-1.5 \le y \le 1.5$, for $C = 0, -\pi/2$, and $\pi/2$, all in the same viewing window. (Experiment with additional values of C.)

(B) Describe how the graph of $y = \cos x$ is changed by changing the values of C in $y = \cos(x + C)$?

66. (A) Graph $y = \sin(x + C)$, $-2\pi \le x \le 2\pi$, $-1.5 \le y \le 1.5$, for $C = 0$, $-\pi/2$, and $\pi/2$, all in the same viewing window. (Experiment with additional values of C.)
(B) Describe how the graph of $y = \sin x$ is changed by changing the values of C in $y = \sin(x + C)$?

67. Try to calculate each of the following on your calculator. Explain the results.
(A) $\sec(\pi/2)$ (B) $\tan(-\pi/2)$ (C) $\cot(-\pi)$

68. Try to calculate each of the following on your calculator. Explain the results.
(A) $\csc \pi$ (B) $\tan(\pi/2)$ (C) $\cot 0$

69. Graph $f(x) = \sin x$ and $g(x) = x$ in the same viewing window $(-1 \le x \le 1, -1 \le y \le 1)$.
(A) What do you observe about the two graphs when x is close to 0, say $-0.5 \le x \le 0.5$?
(B) Complete the table to three decimal places (use the table feature on your graphing utility if it has one):

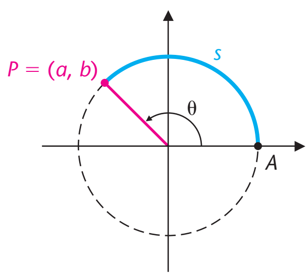
x	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3
$\sin x$							

(In applied mathematics certain derivations, formulas, and calculations are simplified by replacing $\sin x$ with x for small values of $|x|$.)

70. Graph $h(x) = \tan x$ and $g(x) = x$ in the same viewing window $(-1 \le x \le 1, -1 \le y \le 1)$.
(A) What do you observe about the two graphs when x is close to 0, say $-0.5 \le x \le 0.5$?
(B) Complete the table to three decimal places (use the table feature on your graphing utility if it has one):

x	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3
$\tan x$							

(In applied mathematics certain derivations, formulas, and calculations are simplified by replacing $\tan x$ with x for small values of $|x|$.)



71. If the coordinates of A are $(4, 0)$ and arc length s is 7 units, find
(A) The exact radian measure of θ
(B) The coordinates of P to three decimal places

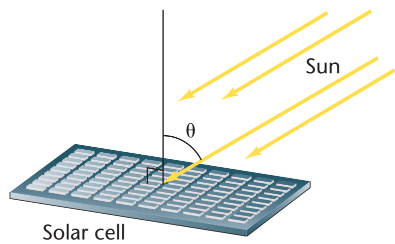
72. If the coordinates of A are $(2, 0)$ and arc length s is 8 units, find
(A) The exact radian measure of θ
(B) The coordinates of P to three decimal places

73. In a rectangular coordinate system, a circle with center at the origin passes through the point $(6\sqrt{3}, 6)$. What is the length of the arc on the circle in quadrant I between the positive horizontal axis and the point $(6\sqrt{3}, 6)$?

74. In a rectangular coordinate system, a circle with center at the origin passes through the point $(2, 2\sqrt{3})$. What is the length of the arc on the circle in quadrant I between the positive horizontal axis and the point $(2, 2\sqrt{3})$?

APPLICATIONS

75. **SOLAR ENERGY** The intensity of light I on a solar cell changes with the angle of the sun and is given by the formula $I = k \cos \theta$, where k is a constant (see the figure). Find light intensity I in terms of k for $\theta = 0°$, $\theta = 30°$, and $\theta = 60°$.

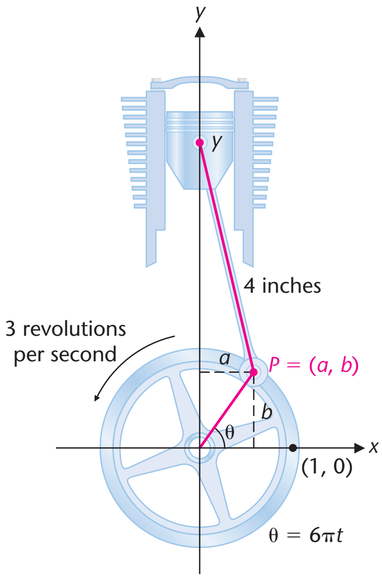


76. **SOLAR ENERGY** Refer to Problem 75. Find light intensity I in terms of k for $\theta = 20°$, $\theta = 50°$, and $\theta = 90°$.

77. **PHYSICS—ENGINEERING** The figure on page 577 illustrates a piston connected to a wheel that turns 3 revolutions per second; hence, the angle θ is being generated at $3(2\pi) = 6\pi$ radians per second, or $\theta = 6\pi t$, where t is time in seconds. If P is at $(1, 0)$ when $t = 0$, show that

$$\begin{aligned} y &= b + \sqrt{4^2 - a^2} \\ &= \sin 6\pi t + \sqrt{16 - (\cos 6\pi t)^2} \end{aligned}$$

for $t \ge 0$.

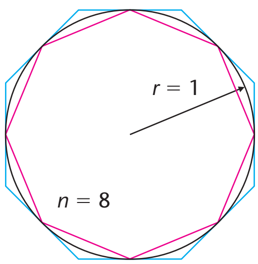


78. PHYSICS—ENGINEERING In Problem 77, find the position of the piston y when $t = 0.2$ second (to three significant digits).

★**79.** GEOMETRY The area of a regular n-sided polygon circumscribed about a circle of radius 1 is given by

$$A = n \tan \frac{180°}{n}$$

(A) Find A for $n = 8$, $n = 100$, $n = 1{,}000$, and $n = 10{,}000$. Compute each to five decimal places.
(B) What number does A seem to approach as $n \to \infty$? (What is the area of a circle with radius 1?)



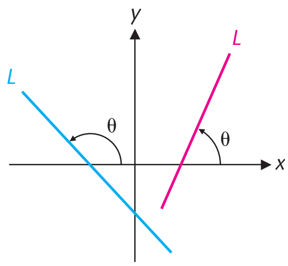
★**80.** GEOMETRY The area of a regular n-sided polygon inscribed in a circle of radius 1 is given by

$$A = \frac{n}{2} \sin \frac{360°}{n}$$

(A) Find A for $n = 8$, $n = 100$, $n = 1{,}000$, and $n = 10{,}000$. Compute each to five decimal places.
(B) What number does A seem to approach as $n \to \infty$? (What is the area of a circle with radius 1?)

81. ANGLE OF INCLINATION Recall (Section 2-3) the **slope** of a nonvertical line passing through points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is given by slope $= m = (y_2 - y_1)/(x_2 - x_1)$. The angle θ that the line L makes with the x axis, $0° \le \theta < 180°$, is called the **angle of inclination** of the line L (see figure). Thus,

$$\text{Slope} = m = \tan \theta,\ 0° \le \theta < 180°$$



(A) Compute the slopes to two decimal places of the lines with angles of inclination 88.7° and 162.3°.
(B) Find the equation of a line passing through $(-4, 5)$ with an angle of inclination 137°. Write the answer in the form $y = mx + b$, with m and b to two decimal places.

82. ANGLE OF INCLINATION Refer to Problem 81.
(A) Compute the slopes to two decimal places of the lines with angles of inclination 5.34° and 92.4°.
(B) Find the equation of a line passing through $(6, -4)$ with an angle of inclination 106°. Write the answer in the form $y = mx + b$, with m and b to two decimal places.

6-5 More General Trigonometric Functions and Models

- Graphs of $y = A \sin Bx$ and $y = A \cos Bx$
- Graphs of $y = A \sin (Bx + C)$ and $y = A \cos (Bx + C)$
- Finding an Equation from the Graph of a Simple Harmonic
- Mathematical Modeling and Data Analysis

Imagine a weight suspended from the ceiling by a spring. If the weight were pulled downward and released, then, assuming no air resistance or friction, it would move up and down with the same frequency and amplitude forever. This idealized motion is an example of **simple harmonic motion.** Simple harmonic motion can be described by functions of the form $y = A \sin (Bx + C)$ or $y = A \cos (Bx + C)$, called **simple harmonics.**

Simple harmonics are extremely important in both pure and applied mathematics. In applied mathematics they are used in the analysis of sound waves, radio waves, X-rays, gamma rays, visible light, infrared radiation, ultraviolet radiation, seismic waves, ocean waves, electric circuits, electric generators, vibrations, bridge and building construction, spring–mass systems, bow waves of boats, sonic booms, and so on. Analysis involving simple harmonics is called *harmonic analysis*.

In Section 6-5 we study properties, graphs, and applications of simple harmonics. A brief review of graph transformations (Section 3-3) should prove helpful.

Graphs of $y = A \sin Bx$ and $y = A \cos Bx$

We visualize the graphs of functions of the form $y = A \sin Bx$ or $y = A \cos Bx$, and determine their zeros and turning points, by understanding how each of the constants A and B transforms the graph of $y = \sin x$ or $y = \cos x$.

EXAMPLE 1 Zeros and Turning Points

Find the zeros and turning points of each function on the interval $[0, 2\pi]$.

(A) $y = \dfrac{1}{2} \sin x$

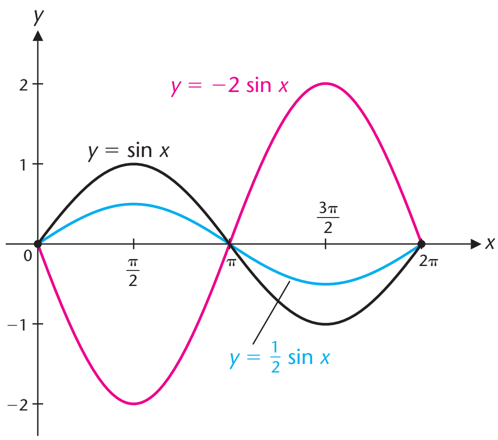
(B) $y = -2 \sin x$

SOLUTIONS

(A) The function $y = \frac{1}{2} \sin x$ is the vertical contraction of $y = \sin x$ that is obtained by multiplying each ordinate value by $\frac{1}{2}$ (Fig. 1). Therefore its zeros on $[0, 2\pi]$

are identical to the zeros of $y = \sin x$, namely, $x = 0$, π, and 2π. Because the turning points of $y = \sin x$ are $(\pi/2, 1)$ and $(3\pi/2, -1)$, the turning points of $y = \frac{1}{2} \sin x$ are $(\pi/2, 1/2)$ and $(3\pi/2, -1/2)$.

(B) The function $y = -2 \sin x$ is the vertical expansion of $y = \sin x$ that is obtained by multiplying each ordinate value by 2, followed by a reflection in the x axis (see Fig. 1). Therefore its zeros on $[0, 2\pi]$ are identical to the zeros of $y = \sin x$, namely $x = 0$, π, and 2π. Because the turning points of $y = \sin x$ are $(\pi/2, 1)$ and $(3\pi/2, -1)$, the turning points of $y = -2 \sin x$ are $(\pi/2, -2)$ and $(3\pi/2, 2)$.



› **Figure 1**

MATCHED PROBLEM 1

Find the zeros and turning points of each function on the interval $[\pi/2, 5\pi/2]$.

(A) $y = -5 \cos x$

(B) $y = \dfrac{1}{3} \cos x$

As Example 1 illustrates, the graph of $y = A \sin x$ can be obtained from the graph of $y = \sin x$ by multiplying each y value of $y = \sin x$ by the constant A. The graph of $y = A \sin x$ still crosses the x axis where the graph of $y = \sin x$ crosses the x axis, because $A \cdot 0 = 0$. Because the maximum value of $\sin x$ is 1, the maximum value of $A \sin x$ is $|A| \cdot 1 = |A|$. The constant $|A|$ is called the **amplitude** of the graph of $y = A \sin x$ and indicates the maximum deviation of the graph of $y = A \sin x$ from the x axis.

The period of $y = A \sin x$ (assuming $A \neq 0$) is the same as the period of $y = \sin x$, namely 2π, because $A \sin (x + 2\pi) = A \sin x$.

EXAMPLE 2 Periods

Find the period of each function.

(A) $y = \sin 2x$ (B) $y = \sin (x/2)$

SOLUTIONS

(A) Because the function $y = \sin x$ has period 2π, the function $y = \sin 2x$ completes one cycle as $2x$ varies from

$$2x = 0 \quad \text{to} \quad 2x = 2\pi$$

or as x varies from

$$x = 0 \quad \text{to} \quad x = \pi \qquad \textbf{Half the period for sin } x.$$

Therefore the period of $y = \sin 2x$ is π (Fig. 2).

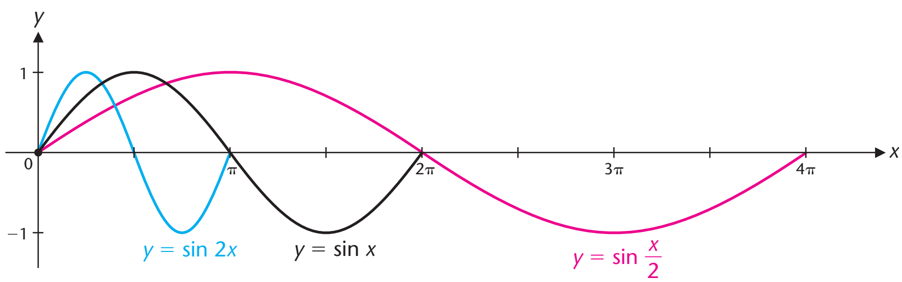
(B) Because the function $y = \sin x$ has period 2π, the function $y = \sin (x/2)$ completes one cycle as $x/2$ varies from

$$\frac{x}{2} = 0 \quad \text{to} \quad \frac{x}{2} = 2\pi$$

or as x varies from

$$x = 0 \quad \text{to} \quad x = 4\pi \qquad \textbf{Double the period for sin } x.$$

Therefore the period of $y = \sin (x/2)$ is 4π (see Fig. 2).



› **Figure 2**

MATCHED PROBLEM 2

Find the period of each function.

(A) $y = \cos (x/10)$ (B) $y = \cos (6\pi x)$

As Example 2 illustrates, the graph of $y = \sin Bx$, for a positive constant B, completes one cycle as Bx varies from

$$Bx = 0 \qquad \text{to} \qquad Bx = 2\pi$$

or as x varies from

$$x = 0 \qquad \text{to} \qquad x = \frac{2\pi}{B}$$

Therefore the period of $y = \sin Bx$ is $\frac{2\pi}{B}$. Note that the amplitude of $y = \sin Bx$ is 1, the same as the amplitude of $y = \sin x$. The effect of the constant B is to compress or stretch the basic sine curve by changing the period of the function, but not its amplitude. A similar analysis applies to $y = \cos Bx$, for $B > 0$, where it can be shown that the period is also $\frac{2\pi}{B}$. We combine and summarize our results on period and amplitude as follows:

› PERIOD AND AMPLITUDE

For $y = A \sin Bx$ or $y = A \cos Bx$, $A \neq 0$, $B > 0$:

$$\text{Amplitude} = |A| \qquad \text{Period} = \frac{2\pi}{B}$$

If $0 < B < 1$, the basic sine or cosine curve is stretched.
If $B > 1$, the basic sine or cosine curve is compressed.

You can either memorize the formula for the period, $\frac{2\pi}{B}$, or use the reasoning we used in deriving the formula. Recall, $\sin Bx$ or $\cos Bx$ completes one cycle as Bx varies from

$$Bx = 0 \qquad \text{to} \qquad Bx = 2\pi$$

that is, as x varies from

$$x = 0 \qquad \text{to} \qquad x = \frac{2\pi}{B}$$

Some prefer to memorize a formula, others a process.

EXAMPLE 3 Amplitude, Period, and Turning Points

Find the amplitude, period, and turning points of $y = -3 \cos (\pi x/2)$ on the interval $[-4, 4]$.

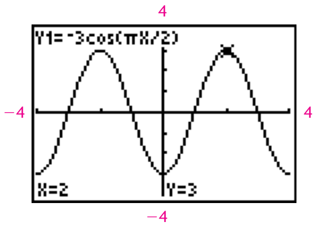


Figure 3

SOLUTION

$$\text{Amplitude} = |-3| = 3 \qquad \text{Period} = \frac{2\pi}{(\pi/2)} = 4$$

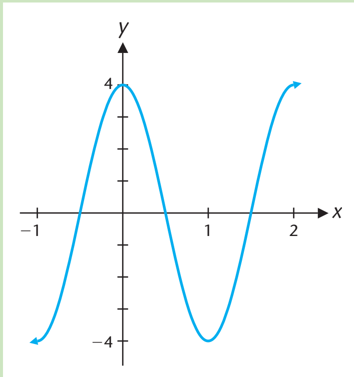
Because $y = \cos x$ has turning points at $x = 0$ and $x = \pm\pi$ (half of a complete cycle), $y = -3 \cos (\pi x/2)$ has turning points at $x = 0$ and $x = \pm 2$. The turning points on the interval $[-4, 4]$ are thus $(-2, 3)$, $(0, -3)$, and $(2, 3)$. These results are confirmed by a graph of $y = -3 \cos (\pi x/2)$ (Fig. 3).

MATCHED PROBLEM 3

Find the amplitude, period, and turning points of $y = \frac{1}{4} \sin (3\pi x)$ on the interval $[0, 1]$.

EXPLORE-DISCUSS 1

Find an equation of the form $y = A \cos Bx$ that produces the following graph.



Is it possible for an equation of the form $y = A \sin Bx$ to produce the same graph? Explain.

Graphs of $y = A \sin (Bx + C)$ and $y = A \cos (Bx + C)$

The graph of $y = A \sin (Bx + C)$ is a horizontal shift of the graph of the function $y = A \sin Bx$. In fact, because the period of the sine function is 2π, $y = A \sin (Bx + C)$ completes one cycle as $Bx + C$ varies from

$$Bx + C = 0 \qquad \text{to} \qquad Bx + C = 2\pi$$

or (solving for x in each equation) as x varies from

$$x = -\overset{\text{Phase shift}}{\frac{C}{B}} \qquad \text{to} \qquad x = -\frac{C}{B} + \overset{\text{Period}}{\frac{2\pi}{B}}$$

We conclude that $y = A \sin (Bx + C)$ has a period of $2\pi/B$, and its graph is the graph of $y = A \sin Bx$ shifted $|-C/B|$ units to the right if $-C/B$ is positive and $|-C/B|$ units to the left if $-C/B$ is negative. The number $-C/B$ is referred to as the **phase shift.**

EXAMPLE 4 Amplitude, Period, Phase Shift, and Zeros

Find the amplitude, period, phase shift, and zeros of $y = \frac{1}{2} \cos (4x - \pi)$, and sketch the graph for $-\pi \le x \le \pi$.

SOLUTION

$$\text{Amplitude} = |A| = \left|\frac{1}{2}\right| = \frac{1}{2}$$

The graph completes one cycle as $4x - \pi$ varies from

$$4x - \pi = 0 \qquad \text{to} \qquad 4x - \pi = 2\pi$$

or as x varies from

$$x = \frac{\pi}{4} \qquad \text{to} \qquad x = \underbrace{\frac{\pi}{4}}_{\text{Phase shift}} + \underbrace{\frac{\pi}{2}}_{\text{Period}} = \frac{3\pi}{4}$$

$$\text{Phase shift} = \frac{\pi}{4} \qquad \text{Period} = \frac{\pi}{2}$$

To sketch the graph, divide the interval $[\pi/4, 3\pi/4]$ into four equal parts and sketch one cycle of $y = \frac{1}{2} \cos (4x - \pi)$. Then extend the graph to cover $[-\pi, \pi]$ (Fig. 4).

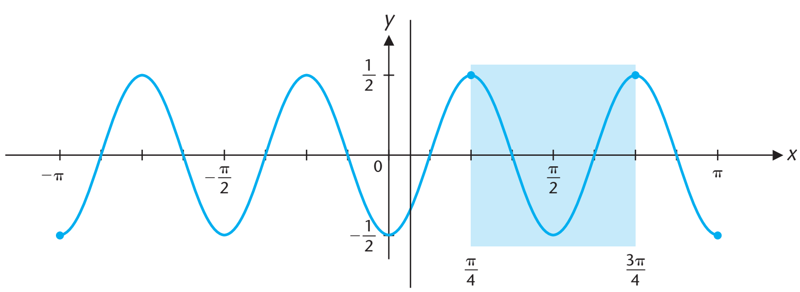


Figure 4
$y = \frac{1}{2} \cos (4x - \pi)$, $-\pi \le x \le \pi$.

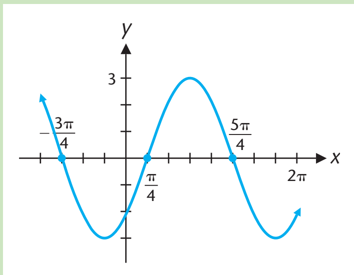
The zeros of $y = \frac{1}{2}\cos(4x - \pi)$ are obtained by shifting the zeros of $y = \frac{1}{2}\cos(4x)$ to the right by $\pi/4$ units. Because $x = \pi/8$ and $x = 3\pi/8$ are zeros of $y = \frac{1}{2}\cos(4x)$, $x = \pi/8 + \pi/4 = 3\pi/8$ and $x = 3\pi/8 + \pi/4 = 5\pi/8$ are zeros of $y = \frac{1}{2}\cos(4x - \pi)$. By periodicity, the zeros of $y = \frac{1}{2}\cos(4x - \pi)$ are $x = 3\pi/8 + k\pi/4$, k any integer, as confirmed by the graph.

MATCHED PROBLEM 4

Find the amplitude, period, phase shift, and zeros of $y = \frac{3}{4}\sin(2x + \pi)$, and sketch the graph for $-\pi \le x \le \pi$.

››› EXPLORE-DISCUSS 2

Find an equation of the form $y = A\sin(Bx + C)$ that produces the following graph.



Is it possible for an equation of the form $y = A\cos(Bx + C)$ to produce the same graph? Explain.

The graphs of $y = A\sin(Bx + C) + k$ and $y = A\cos(Bx + C) + k$ are vertical shifts (up k units if $k > 0$, down k units if $k < 0$) of the graphs of $y = A\sin(Bx + C)$ and $y = A\cos(Bx + C)$, respectively.

Because $y = \sec x$ and $y = \csc x$ are unbounded functions, amplitude is not defined for functions of the form $y = A\sec(Bx + C)$ and $y = A\csc(Bx + C)$. However, because both the secant and cosecant functions have period 2π, the functions $y = A\csc(Bx + C)$ and $y = A\sec(Bx + C)$ have period $2\pi/B$ and phase shift $-C/B$.

Because $y = \tan x$ and $y = \cot x$ are unbounded functions, amplitude is not defined for functions of the form $y = A\tan(Bx + C)$ or $y = A\cot(Bx + C)$. The tangent and cotangent functions both have period π, so the functions $y = A\tan(Bx + C)$ and $y = A\cot(Bx + C)$ have period π/B and phase shift $-C/B$.

Our results on amplitude, period, and phase shift are summarized in the following box.

AMPLITUDE, PERIOD, AND PHASE SHIFT

Let A, B, C be constants such that $A \neq 0$ and $B > 0$.
For $y = A \sin (Bx + C)$ and $y = A \cos (Bx + C)$:

$$\text{Amplitude} = |A| \qquad \text{Period} = \frac{2\pi}{B} \qquad \text{Phase shift} = \frac{-C}{B}$$

For $y = A \sec (Bx + C)$ and $y = A \csc (Bx + C)$:

$$\text{Period} = \frac{2\pi}{B} \qquad \text{Phase shift} = \frac{-C}{B}$$

For $y = A \tan (Bx + C)$ and $y = A \cot (Bx + C)$:

$$\text{Period} = \frac{\pi}{B} \qquad \text{Phase shift} = \frac{-C}{B}$$

Note: Amplitude is not defined for the secant, cosecant, tangent, and cotangent functions, all of which are unbounded.

Finding an Equation from the Graph of a Simple Harmonic

Given the graph of a simple harmonic, we wish to find an equation of the form $y = A \sin (Bx + C)$ or $y = A \cos (Bx + C)$ that produces the graph. Example 5 illustrates the process.

EXAMPLE 5

Finding an Equation of a Simple Harmonic Graph

Graph $y_1 = 3 \sin x + 4 \cos x$ using a graphing calculator, and find an equation of the form $y_2 = A \sin (Bx + C)$ that has the same graph as y_1. Find A and B exactly and C to three decimal places.

SOLUTION

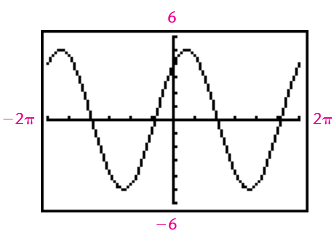


Figure 5
$y_1 = 3 \sin x + 4 \cos x$.

The graph of y_1 is shown in Figure 5. The graph appears to be a sine curve shifted to the left. The amplitude and period appear to be 5 and 2π, respectively. (We will assume this for now and check it at the end.) Thus, $A = 5$, and because $P = 2\pi/B$, then $B = 2\pi/P = 2\pi/2\pi = 1$. Using a graphing calculator, we find that the x intercept closest to the origin, to three decimal places, is -0.927. To find C, substitute $B = 1$ and $x = -0.927$ into the phase-shift formula $x = -C/B$ and solve for C:

$$x = -\frac{C}{B} \qquad \text{Substitute } x = -0.927,\ B = 1.$$

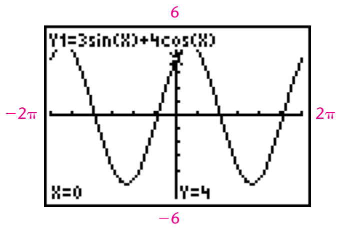
$$-0.927 = -\frac{C}{1} \qquad \text{Solve for } C.$$

$$C = 0.927$$

We now have the equation we are looking for:

$$y_2 = 5 \sin (x + 0.927)$$

CHECK Graph y_1 and y_2 in the same viewing window. If the graphs are the same, it appears that only one graph is drawn—the second graph is drawn over the first. To check further that the graphs are the same, use TRACE and switch back and forth between y_1 and y_2 at different values of x. Figure 6 shows a comparison at $x = 0$ (both graphs appear in the same viewing window).



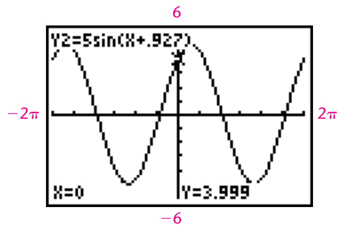


Figure 6

MATCHED PROBLEM 5

Graph $y_1 = 4 \sin x - 3 \cos x$ using a graphing calculator, and find an equation of the form $y_2 = A \sin (Bx + C)$ that has the same graph as y_1. (Find the x intercept closest to the origin to three decimal places.)

Mathematical Modeling and Data Analysis

The polynomial, exponential, and logarithmic functions studied in Chapters 4 and 5 are not suitable for modeling periodic phenomena. Instead, when given a data set that indicates periodic behavior, we use a technique called **sinusoidal regression** to model the data by a function of the form $f(x) = A \sin (Bx + C) + k$.

EXAMPLE 6 Temperature Variation

The monthly average high temperatures in Fairbanks, Alaska, are given in Table 1. A sinusoidal model for the data is given by

$$y = 37.4 \sin (0.523x - 1.93) + 37.2$$

where x is time in months ($x = 1$ represents January 15, $x = 2$ represents February 15, etc.) and y is temperature in degrees Fahrenheit. Use the sinusoidal regression function to estimate the average high temperature on April 1 to one decimal place.

Table 1 Temperatures in Fairbanks, Alaska

Month	1	2	3	4	5	6	7	8	9	10	11	12
Average High (°F)	0	8	25	44	61	71	73	66	54	31	11	3
Average Low (°F)	−19	−15	−3	20	37	49	52	46	35	16	−7	−15

SOLUTION

To estimate the average high temperature on April 1 we substitute $x = 3.5$:

$$y = 37.4 \sin (0.523 \cdot 3.5 - 1.93) + 37.2 \approx 33.5°$$

Technology Connections

Figure 7 shows the details of constructing the sinusoidal model of Example 6 on a graphing calculator. To observe the cyclical behavior of the data, we enter the average high temperatures for two consecutive years, from $x = 1$ to $x = 24$. The data, the sinusoidal regression function, and a plot of the data and graph of the regression function are shown in Figure 7. To estimate the average high temperature on April 1, we let $x = 3.5$ [Fig. 7(c)]. Note the slight discrepancy between the estimated high temperature (33.5°) of Example 6, and the value given in Figure 7(c) (approximately 33.6°), due to rounding the coefficients of the regression equation to three significant digits.

(a) Entering the data

(b) Finding the model

(c) Graphing the data and the model

› Figure 7

MATCHED PROBLEM 6

The monthly average low temperatures in Fairbanks, Alaska, are given in Table 1. A sinusoidal model for the data is given by

$$y = 36.7 \sin (0.524x - 2.05) + 16.4$$

where x is the time in months ($x = 1$ represents January 15, $x = 2$ represents February 15, etc.) and y is temperature in degrees Fahrenheit. Use the sinusoidal regression function to estimate the average low temperature on April 1 to one decimal place.

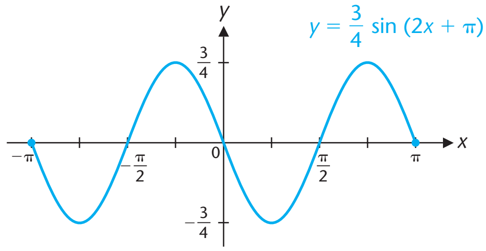
ANSWERS TO MATCHED PROBLEMS

1. (A) Zeros: $\pi/2$, $3\pi/2$, $5\pi/2$; turning points: $(\pi, 5)$, $(2\pi, -5)$
(B) Zeros: $\pi/2$, $3\pi/2$, $5\pi/2$; turning points: $(\pi, -1/3)$, $(2\pi, 1/3)$

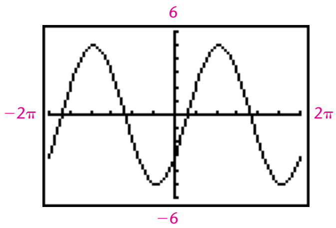
2. (A) 20π (B) $1/3$

3. Amplitude: $1/4$; period: $2/3$; turning points: $(1/6, 1/4)$, $(1/2, -1/4)$, $(5/6, 1/4)$

4. Amplitude: $3/4$; period: π; phase shift: $-\pi/2$; zeros: $k\pi/2$, k any integer



5. $y_2 = 5 \sin (x - 0.644)$



6. 8.53°F

6-5 Exercises

In Problems 1–12, find the amplitude (if applicable) and period.

1. $y = 3 \sin x$

2. $y = \frac{1}{4} \cos x$

3. $y = -\frac{1}{2} \cos x$

4. $y = -2 \sin x$

5. $y = \sin 3x$

6. $y = \cos 2x$

7. $y = 2 \cot 4x$

8. $y = 3 \tan 2x$

9. $y = -\frac{1}{4} \tan 8\pi x$

10. $y = -\frac{1}{2} \cot 2\pi x$

11. $y = \csc (x/2)$

12. $y = \sec \pi x$

In Problems 13–16, find the amplitude (if applicable), the period, and all zeros in the given interval.

13. $y = \sin \pi x,\ -2 \le x \le 2$

14. $y = \cos \pi x,\ -2 \le x \le 2$

15. $y = \frac{1}{2} \cot (x/2),\ 0 < x < 4\pi$

16. $y = \frac{1}{2} \tan (x/2),\ -\pi < x < 3\pi$

In Problems 17–20, find the amplitude (if applicable), the period, and all turning points in the given interval.

17. $y = 3 \cos 2x,\ -\pi \le x \le \pi$

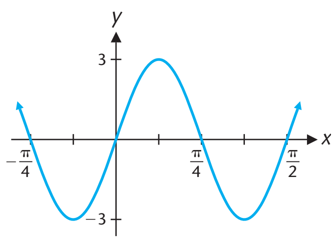
18. $y = 2 \sin 4x,\ -\pi \le x \le \pi$

19. $y = 2 \sec \pi x,\ -1 \le x \le 3$

20. $y = 2 \csc (x/2),\ 0 < x < 8\pi$

In Problems 21–24, find the equation of the form $y = A \sin Bx$ that produces the graph shown.

21.



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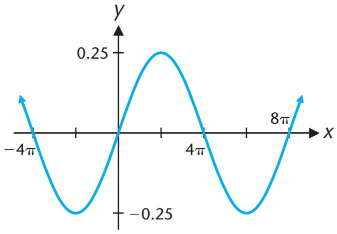

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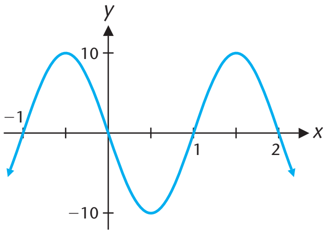

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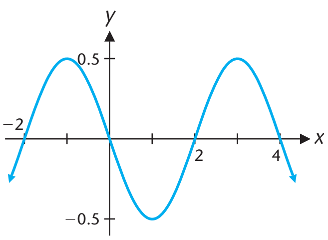
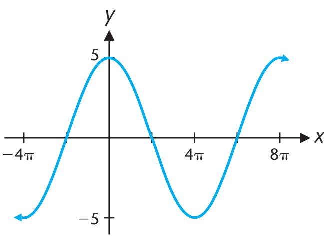


In Problems 25–28, find the equation of the form $y = A \cos Bx$ that produces the graph shown.

25.



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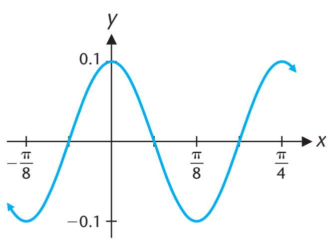

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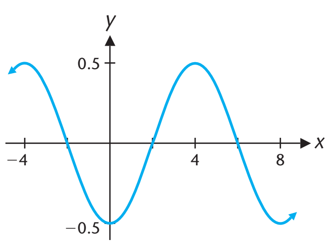
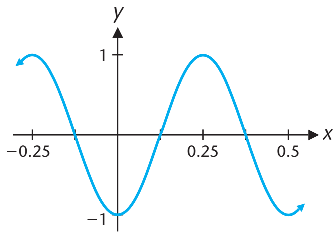


28.



In Problems 29–44, find the amplitude (if applicable), period, and phase shift, then graph each function.

29. $y = 4 \cos x,\ 0 \le x \le 4\pi$

30. $y = 5 \sin x,\ 0 \le x \le 4\pi$

31. $y = \frac{1}{2} \sin (x + \pi/4),\ -2\pi \le x \le 2\pi$

32. $y = \frac{1}{3} \cos (x - \pi/4),\ -2\pi \le x \le 2\pi$

33. $y = \cot (x - \pi/6),\ -\pi \le x \le \pi$

34. $y = \tan (x + \pi/3),\ -\pi \le x \le \pi$

35. $y = 3 \tan 2x,\ 0 \le x \le 2\pi$

36. $y = -4 \cot 3x,\ -\pi/2 \le x \le \pi/2$

37. $y = 2\pi \sin (\pi x/2),\ 0 \le x \le 12$

38. $y = \pi \cos (\pi x/4),\ 0 \le x \le 12$

39. $y = -3 \sin [2\pi(x + \frac{1}{2})],\ -1 \le x \le 2$

40. $y = -2 \cos [\pi(x - 1)],\ -1 \le x \le 2$

41. $y = \sec (x + \pi),\ -\pi \le x \le \pi$

42. $y = \csc (x - \pi/2),\ -\pi \le x \le \pi$

43. $y = 10 \csc \pi x,\ 0 < x < 3$

44. $y = 8 \sec 2\pi x,\ 0 \le x \le 3$

In Problems 45–56, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

45. The graph of $y = A \sin Bx$ passes through the origin.

46. The graph of every simple harmonic passes through the origin.

47. Every simple harmonic is either even or odd.

48. The function $y = A \cos Bx$ is even.

49. Every simple harmonic is periodic.

50. Every simple harmonic is periodic with period 2π.

51. Every simple harmonic is bounded.

52. The amplitude of every simple harmonic is greater than its period.

53. If f is a simple harmonic, then the function g defined by $g(x) = f(3x)$ is a simple harmonic.

54. If f is a simple harmonic, then the function h defined by $h(x) = f(x + 3)$ is a simple harmonic.

55. If f is a simple harmonic, then the function j defined by $j(x) = 3 - f(x)$ is a simple harmonic.

56. If f is a simple harmonic, then the function k defined by $k(x) = 3f(x)$ is a simple harmonic.

Graph each function in Problems 57–60. (Select the dimensions of each viewing window so that at least two periods are visible.) Find an equation of the form $y = k + A \sin Bx$ or $y = k + A \cos Bx$ that has the same graph as the given equation. (These problems suggest the existence of further identities in addition to the basic identities discussed in Section 6-4.)

57. $y = \cos^2 x - \sin^2 x$

58. $y = \sin x \cos x$

59. $y = 2 \sin^2 x$

60. $y = 2 \cos^2 x$

In Problems 61–68, graph at least two cycles of the given equation in a graphing calculator, then find an equation of the form $y = A \tan Bx$, $y = A \cot Bx$, $y = A \sec Bx$, or $y = A \csc Bx$ that has the same graph. (These problems suggest additional identities beyond those discussed in Section 6-4. Additional identities are discussed in detail in Chapter 7.)

61. $y = \cot x - \tan x$

62. $y = \cot x + \tan x$

63. $y = \csc x + \cot x$

64. $y = \csc x - \cot x$

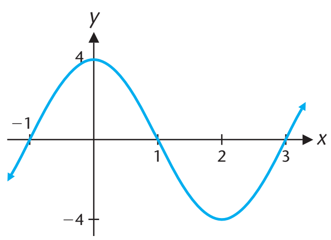
65. $y = \sin 3x + \cos 3x \cot 3x$

66. $y = \cos 2x + \sin 2x \tan 2x$

67. $y = \dfrac{\sin 4x}{1 + \cos 4x}$

68. $y = \dfrac{\sin 6x}{1 - \cos 6x}$

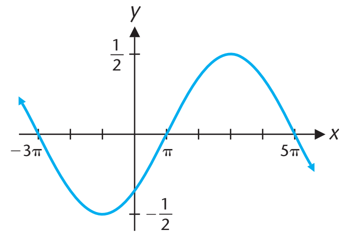
Problems 69 and 70 refer to the following graph:



69. If the graph is a graph of an equation of the form $y = A \sin (Bx + C)$, $0 < -C/B < 2$, find the equation.

70. If the graph is a graph of an equation of the form $y = A \sin (Bx + C)$, $-2 < -C/B < 0$, find the equation.

Problems 71 and 72 refer to the following graph:



71. If the graph is a graph of an equation of the form $y = A \cos (Bx + C)$, $0 < -C/B < 4\pi$, find the equation.

72. If the graph is a graph of an equation of the form $y = A \cos (Bx + C)$, $-2\pi < -C/B < 0$, find the equation.

In Problems 73–76, state the amplitude, period, and phase shift of each function and sketch a graph of the function with the aid of a graphing calculator.

73. $y = 3.5 \sin\left[\dfrac{\pi}{2}(t + 0.5)\right]$, $0 \le t \le 10$

74. $y = 5.4 \sin\left[\dfrac{\pi}{2.5}(t - 1)\right]$, $0 \le t \le 6$

75. $y = 50 \cos [2\pi(t - 0.25)]$, $0 \le t \le 2$

76. $y = 25 \cos [5\pi(t - 0.1)]$, $0 \le t \le 2$

In Problems 77–82, graph each equation. (Select the dimensions of each viewing window so that at least two periods are visible.) Find an equation of the form $y = A \sin (Bx + C)$ that has the same graph as the given equation. Find A and B exactly and C to three decimal places. Use the x intercept closest to the origin as the phase shift.

77. $y = \sqrt{2} \sin x + \sqrt{2} \cos x$

78. $y = \sqrt{2} \sin x - \sqrt{2} \cos x$

79. $y = \sqrt{3} \sin x - \cos x$

80. $y = \sin x + \sqrt{3} \cos x$

81. $y = 4.8 \sin 2x - 1.4 \cos 2x$

82. $y = 1.4 \sin 2x + 4.8 \cos 2x$

Problems 83–88 illustrate combinations of functions that occur in harmonic analysis applications. Graph parts A, B, and C of each problem in the same viewing window. In Problems 83–86, what is happening to the amplitude of the function in part C? Give an example of a physical phenomenon that might be modeled by a similar function.

83. $0 \le x \le 16$

(A) $y = \dfrac{1}{x}$ (B) $y = -\dfrac{1}{x}$ (C) $y = \dfrac{1}{x} \sin \dfrac{\pi}{2} x$

84. $0 \le x \le 10$

(A) $y = \dfrac{2}{x}$ (B) $y = -\dfrac{2}{x}$ (C) $y = \dfrac{2}{x} \cos \pi x$

85. $0 \le x \le 10$

(A) $y = x$ (B) $y = -x$ (C) $y = x \sin \dfrac{\pi}{2} x$

86. $0 \le x \le 10$

(A) $y = \dfrac{x}{2}$ (B) $y = -\dfrac{x}{2}$ (C) $y = \dfrac{x}{2} \cos \pi x$

87. $0 \le x \le 2\pi$

(A) $y = \sin x$ (B) $y = \sin x + \dfrac{\sin 3x}{3}$

(C) $y = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5}$

88. $0 \le x \le 4$

(A) $y = \sin \pi x$ (B) $y = \sin \pi x + \dfrac{\sin 2\pi x}{2}$

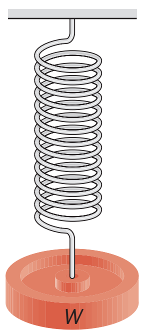
(C) $y = \sin \pi x + \dfrac{\sin 2\pi x}{2} + \dfrac{\sin 3\pi x}{3}$

APPLICATIONS

89. SPRING-MASS SYSTEM A 6-pound weight hanging from the end of a spring is pulled $\frac{1}{3}$ foot below the equilibrium position and then released (see figure). If air resistance and friction are neglected, the distance x that the weight is from the equilibrium position relative to time t (in seconds) is given by

$$x = \tfrac{1}{3} \cos 8t$$

State the period P and amplitude A of this function, and graph it for $0 \le t \le \pi$.



90. ELECTRICAL CIRCUIT An alternating current generator generates a current given by

$$I = 30 \sin 120t$$

where t is time in seconds. What are the amplitude A and period P of this function? What is the frequency of the current; that is, how many cycles (periods) will be completed in 1 second?

★ **91. SPRING-MASS SYSTEM** Assume the motion of the weight in Problem 89 has an amplitude of 8 inches and a period of 0.5 second, and that its position when $t = 0$ is 8 inches below its position at rest (displacement above rest position is positive and below is negative). Find an equation of the form $y = A \cos Bt$ that describes the motion at any time $t \ge 0$. (Neglect any damping forces—that is, friction and air resistance.)

★ **92. ELECTRICAL CIRCUIT** If the voltage E in an electrical circuit has an amplitude of 110 volts and a period of $\frac{1}{60}$ second, and if $E = 110$ volts when $t = 0$ seconds, find an equation of the form $E = A \cos Bt$ that gives the voltage at any time $t \ge 0$.

93. POLLUTION The amount of sulfur dioxide pollutant from heating fuels released in the atmosphere in a city varies seasonally. Suppose the number of tons of pollutant released into the atmosphere during the nth week after January 1 for a particular city is given by

$$A(n) = 1.5 + \cos \frac{n\pi}{26} \qquad 0 \le n \le 104$$

Graph the function over the indicated interval and describe what the graph shows.

94. MEDICINE A seated normal adult breathes in and exhales about 0.82 liter of air every 4.00 seconds. The volume of air in the lungs t seconds after exhaling is approximately

$$V(t) = 0.45 - 0.37 \cos \frac{\pi t}{2} \qquad 0 \le t \le 8$$

Graph the function over the indicated interval and describe what the graph shows.

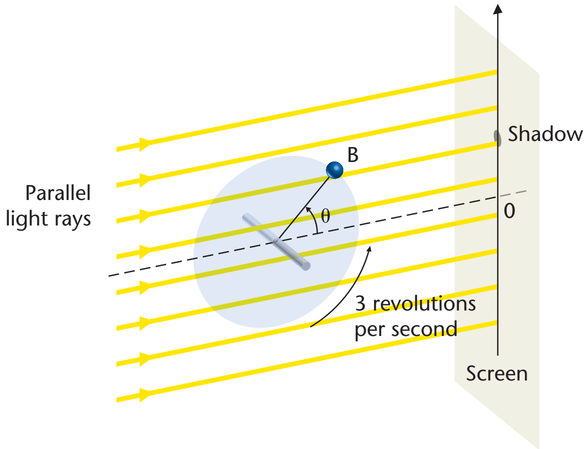
95. ELECTRICAL CIRCUIT The current in an electrical circuit is given by $I = 15 \cos (120\pi t + \pi/2)$, $0 \le t \le \frac{2}{60}$, where I is measured in amperes. State the amplitude A, period P, and phase shift. Graph the equation.

96. ELECTRICAL CIRCUIT The current in an electrical circuit is given by $I = 30 \cos (120\pi t - \pi)$, $0 \le t \le \frac{3}{60}$, where I is measured in amperes. State the amplitude A, period P, and phase shift. Graph the equation.

97. PHYSICS—ENGINEERING The thin, plastic disk shown in the figure on page 592 is rotated at 3 revolutions per second, starting at $\theta = 0$ (thus at the end of t seconds, $\theta = 6\pi t$—Why?). If the disk has a radius of 3, show that the position of the shadow on the y scale from the small steel ball B is given by

$$y = 3\sin 6\pi t$$

Graph this equation for $0 \le t \le 1$.

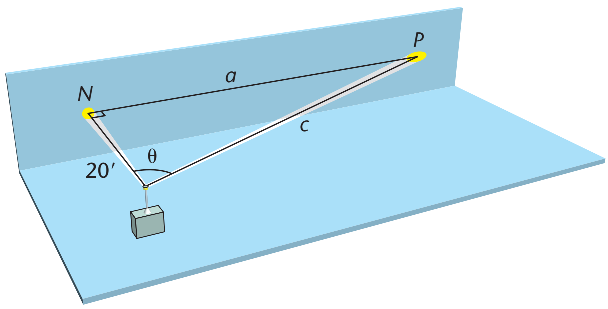


98. **PHYSICS—ENGINEERING** If in Problem 97 the disk started rotating at $\theta = \pi/2$, show that the position of the shadow at time t (in seconds) is given by

$$y = 3 \sin\left(6\pi t + \frac{\pi}{2}\right)$$

Graph this equation for $0 \le t \le 1$.

★ **99.** A beacon light 20 feet from a wall rotates clockwise at the rate of 1/4 revolution per second (rps) (see the figure), thus, $\theta = \pi t/2$.
(A) Start counting time in seconds when the light spot is at N and write an equation for the length c of the light beam in terms of t.
(B) Graph the equation found in part A for the time interval [0, 1].
(C) Describe what happens to the length c of the light beam as t goes from 0 to 1.



★ **100.** Refer to Problem 99.
(A) Write an equation for the distance a the light spot travels along the wall in terms of time t.
(B) Graph the equation found in part A for the time interval [0, 1].
(C) Describe what happens to the distance a along the wall as t goes from 0 to 1.

101. **MODELING SUNSET TIMES** Sunset times for the fifth of each month over a period of 1 year were taken from a tide booklet for the San Francisco Bay to form Table 2. Daylight savings time was ignored and the times are for a 24-hour clock starting at midnight.
(A) Using 1 month as the basic unit of time, enter the data for a 2-year period in your graphing calculator and produce a scatter plot in the viewing window. Before entering Table 2 data into your graphing calculator, convert sunset times from hours and minutes to decimal hours rounded to two decimal places. Choose $15 \le y \le 20$ for the viewing window.
(B) It appears that a sine curve of the form

$$y = k + A \sin(Bx + C)$$

will closely model these data. The constants k, A, and B are easily determined from Table 2 as follows: $A = (\max y - \min y)/2$, $B = 2\pi/\text{Period}$, and $k = \min y + A$. To estimate C, visually estimate to one decimal place the smallest positive phase shift from the plot in part A. After determining A, B, k, and C, write the resulting equation. (Your value of C may differ slightly from the answer in the back of the book.)
(C) Plot the results of parts A and B in the same viewing window. (An improved fit may result by adjusting your value of C slightly.)
(D) If your graphing utility has a sinusoidal regression feature, check your results from parts B and C by finding and plotting the regression equation.

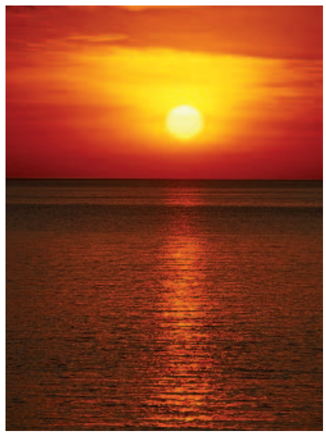


Table 2

x (months)	1	2	3	4	5	6	7	8	9	10	11	12
y (sunset)*	17:05	17:38	18:07	18:36	19:04	19:29	19:35	19:15	18:34	17:47	17:07	16:51

*Time on a 24-hr clock, starting at midnight.

102. MODELING TEMPERATURE VARIATION The 30-year average monthly temperature, °F, for each month of the year for Washington, D.C., is given in Table 3 (*World Almanac*).
(A) Using 1 month as the basic unit of time, enter the data for a 2-year period in your graphing calculator and produce a scatter plot in the viewing window. Choose $0 \le y \le 80$ for the viewing window.
(B) It appears that a sine curve of the form

$$y = k + A \sin (Bx + C)$$

will closely model these data. The constants k, A, and B are easily determined from Table 3 as follows: $A = (\max y - \min y)/2$, $B = 2\pi/\text{Period}$, and $k = \min y + A$. To estimate C, visually estimate to one decimal place the smallest positive phase shift from the plot in part A. After determining A, B, k, and C, write the resulting equation.
(C) Plot the results of parts A and B in the same viewing window. (An improved fit may result by adjusting your value of C slightly.)
(D) If your graphing calculator has a sinusoidal regression feature, check your results from parts B and C by finding and plotting the regression equation.

Table 3

x (months)	1	2	3	4	5	6	7	8	9	10	11	12
y (temp.)	31	34	43	53	62	71	76	74	67	55	45	35

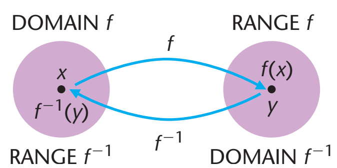
6-6 Inverse Trigonometric Functions

- Inverse Sine Function
- Inverse Cosine Function
- Inverse Tangent Function
- Summary
- Inverse Cotangent, Secant, and Cosecant Functions (Optional)

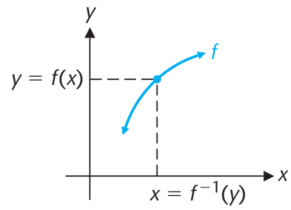
A brief review of the general concept of inverse functions discussed in Section 3-6 should prove helpful before proceeding with Section 6-6. In the box we restate a few important facts about inverse functions from Section 3-6.

FACTS ABOUT INVERSE FUNCTIONS

For a one-to-one function f and its inverse f^{-1}:

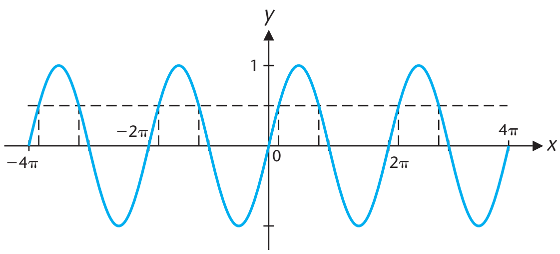
1. If (a, b) is an element of f, then (b, a) is an element of f^{-1}, and conversely.
2. Range of f = Domain of f^{-1}
   Domain of f = Range of f^{-1}
3. 

4. If $x = f^{-1}(y)$, then $y = f(x)$ for y in the domain of f^{-1} and x in the domain of f, and conversely.



5. $f(f^{-1}(y)) = y$ for y in the domain of f^{-1}
$f^{-1}(f(x)) = x$ for x in the domain of f

All trigonometric functions are periodic; hence, each range value can be associated with infinitely many domain values (Fig. 1). As a result, no trigonometric function is one-to-one, so, strictly speaking, no trigonometric function has an inverse. However, we can restrict the domain of each function so that it is one-to-one over the restricted domain. Then, for this restricted domain, an inverse function is guaranteed.



› **Figure 1**
$y = \sin x$ is not one-to-one over $(-\infty, \infty)$.

Inverse trigonometric functions represent another group of basic functions that are added to our library of elementary functions. These functions are used in many applications and mathematical developments, and will be particularly useful to us when we solve trigonometric equations in Section 7-5.

› Inverse Sine Function

How can the domain of the sine function be restricted so that it is one-to-one? This can be done in infinitely many ways. A fairly natural and generally accepted way is illustrated in Figure 2.

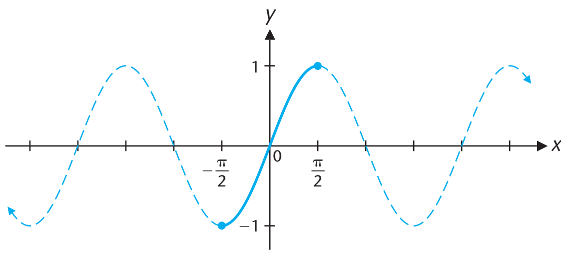


Figure 2
$y = \sin x$ is one-to-one over $[-\pi/2, \pi/2]$.

If the domain of the sine function is restricted to the interval $[-\pi/2, \pi/2]$, we see that the restricted function passes the horizontal line test (Section 3-6) and thus is one-to-one. Note that each range value from -1 to 1 is assumed exactly once as x moves from $-\pi/2$ to $\pi/2$. We use this restricted sine function to define the *inverse sine function.*

DEFINITION 1 Inverse Sine Function

The **inverse sine function,** denoted by $\sin^{-1}$ or arcsin, is defined as the inverse of the restricted sine function $y = \sin x$, $-\pi/2 \le x \le \pi/2$. Thus,

$$y = \sin^{-1} x \qquad \text{and} \qquad y = \arcsin x$$

are equivalent to

$$\sin y = x \qquad \text{where} \qquad -\pi/2 \le y \le \pi/2, \ -1 \le x \le 1$$

In words, the inverse sine of x, or the arcsine of x, is the number or angle y, $-\pi/2 \le y \le \pi/2$, whose sine is x.

To graph $y = \sin^{-1} x$, take each point on the graph of the restricted sine function and reverse the order of the coordinates. For example, because $(-\pi/2, -1)$, $(0, 0)$, and $(\pi/2, 1)$ are on the graph of the restricted sine function [Fig. 3(a)], then $(-1, -\pi/2)$, $(0, 0)$, and $(1, \pi/2)$ are on the graph of the inverse sine function, as shown in Figure 3(b). Using these three points provides us with a quick way of sketching the graph of the inverse sine function. A more accurate graph can be obtained by using a calculator.

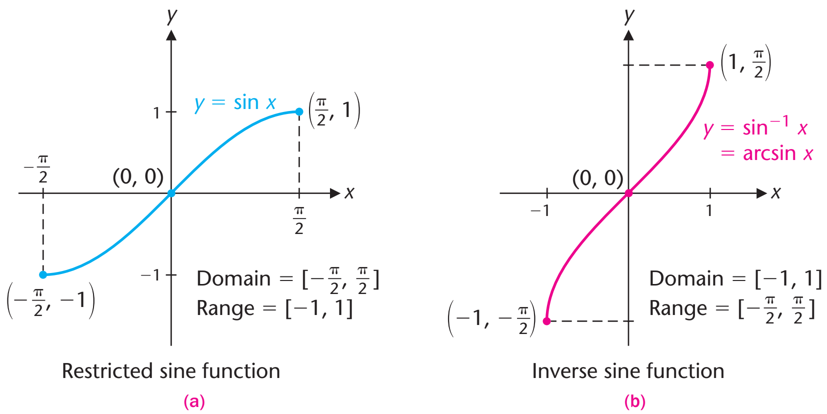


Figure 3 Inverse sine function.

EXPLORE-DISCUSS 1

A graphing calculator produced the graph in Figure 4 for $y_1 = \sin^{-1} x$, $-2 \le x \le 2$, and $-2 \le y \le 2$. Explain why there are no parts of the graph on the intervals $[-2, -1)$ and $(1, 2]$.

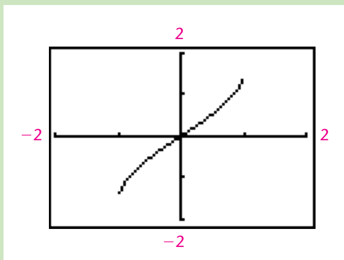


Figure 4

We state the important sine–inverse sine identities that follow from the general properties of inverse functions given in the box at the beginning of this section.

SINE–INVERSE SINE IDENTITIES

$\sin(\sin^{-1} x) = x$	$-1 \le x \le 1$	$f(f^{-1}(x)) = x$
$\sin^{-1}(\sin x) = x$	$-\pi/2 \le x \le \pi/2$	$f^{-1}(f(x)) = x$

$\sin(\sin^{-1} 0.7) = 0.7$ $\quad\sin(\sin^{-1} 1.3) \ne 1.3$

$\sin^{-1}[\sin(-1.2)] = -1.2$ $\quad\sin^{-1}[\sin(-2)] \ne -2$

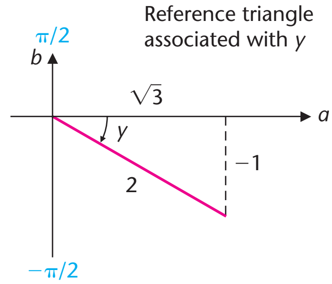
[*Note:* The number 1.3 is not in the domain of the inverse sine function, and -2 is not in the restricted domain of the sine function. Try calculating all these examples with your calculator and see what happens!]

EXAMPLE 1 Exact Values

Find exact values without using a calculator.

(A) $\arcsin\left(-\frac{1}{2}\right)$ (B) $\sin^{-1}(\sin 1.2)$ (C) $\cos\left[\sin^{-1}\left(\frac{2}{3}\right)\right]$

SOLUTIONS



(A) $y = \arcsin\left(-\frac{1}{2}\right)$ is equivalent to

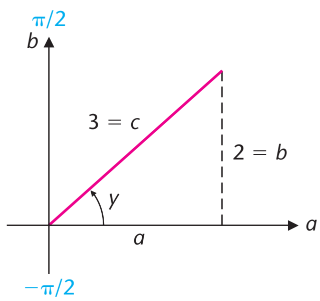
$$\sin y = -\tfrac{1}{2} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$y = -\frac{\pi}{6} = \arcsin\left(-\tfrac{1}{2}\right)$$

[*Note:* $y \ne 11\pi/6$, even though $\sin(11\pi/6) = -\frac{1}{2}$ because y must be between $-\pi/2$ and $\pi/2$, inclusive.]

(B) $\sin^{-1}(\sin 1.2) = 1.2$ Sine–inverse sine identity, because $-\pi/2 \le 1.2 \le \pi/2$

(C) Let $y = \sin^{-1}\left(\frac{2}{3}\right)$; then $\sin y = \frac{2}{3}$, $-\pi/2 \le y \le \pi/2$. Draw the reference triangle associated with y. Then $\cos y = \cos\left[\sin^{-1}\left(\frac{2}{3}\right)\right]$ can be determined directly from the triangle (after finding the third side) without actually finding y.



$$a^2 + b^2 = c^2$$
$$a = \sqrt{3^2 - 2^2} \quad \text{Because } a > 0 \text{ in quadrant I}$$
$$= \sqrt{5}$$

Thus, $\cos\left[\sin^{-1}\left(\frac{2}{3}\right)\right] = \cos y = \sqrt{5}/3$.

MATCHED PROBLEM 1

Find exact values without using a calculator.

(A) $\arcsin(\sqrt{2}/2)$ (B) $\sin\left[\sin^{-1}(-0.4)\right]$ (C) $\tan\left[\sin^{-1}(-1/\sqrt{5})\right]$

EXAMPLE 2 Calculator Values

Find to four significant digits using a calculator.

(A) $\arcsin(-0.3042)$ (B) $\sin^{-1} 1.357$ (C) $\cot\left[\sin^{-1}(-0.1087)\right]$

SOLUTIONS

The function keys used to represent inverse trigonometric functions vary among different brands of calculators, so read the user's manual for your calculator. Set your calculator in radian mode and follow your manual for key sequencing.

(A) $\arcsin(-0.3042) = -0.3091$

(B) $\sin^{-1} 1.357 = \text{Error}$ 1.357 is not in the domain of $\sin^{-1}$

(C) $\cot[\sin^{-1}(-0.1087)] = -9.145$

MATCHED PROBLEM 2

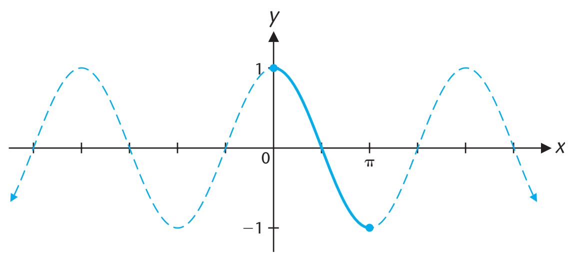
Find to four significant digits using a calculator.

(A) $\sin^{-1} 0.2903$ (B) $\arcsin(-2.305)$ (C) $\cot[\sin^{-1}(-0.3446)]$

Inverse Cosine Function

To restrict the cosine function so that it becomes one-to-one, we choose the interval $[0, \pi]$. Over this interval the restricted function passes the horizontal line test, and each range value is assumed exactly once as x moves from 0 to π (Fig. 5). We use this restricted cosine function to define the *inverse cosine function.*

Figure 5 $y = \cos x$ is one-to-one over $[0, \pi]$.



DEFINITION 2 Inverse Cosine Function

The **inverse cosine function,** denoted by $\cos^{-1}$ or arccos, is defined as the inverse of the restricted cosine function $y = \cos x$, $0 \le x \le \pi$. Thus,

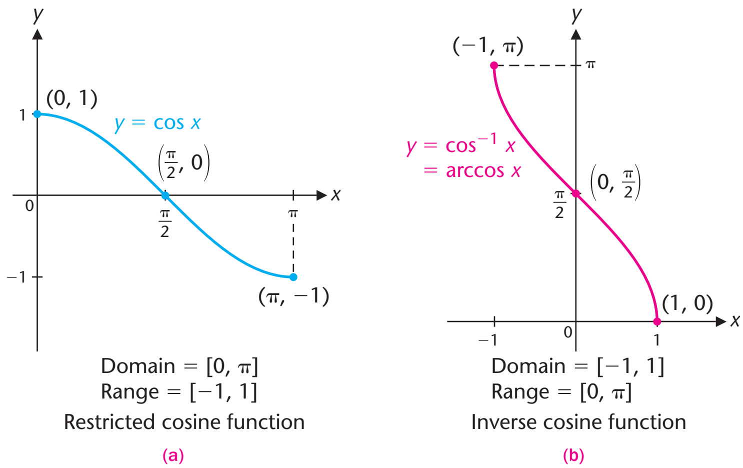
$$y = \cos^{-1} x \quad \text{and} \quad y = \arccos x$$

are equivalent to

$$\cos y = x \quad \text{where} \quad 0 \le y \le \pi, \ -1 \le x \le 1$$

In words, the inverse cosine of x, or the arccosine of x, is the number or angle y, $0 \le y \le \pi$, whose cosine is x.

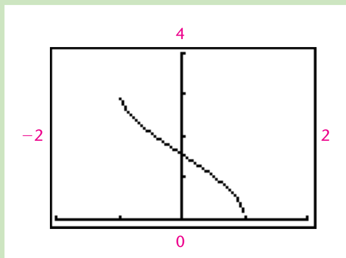
Figure 6 compares the graphs of the restricted cosine function and its inverse. Notice that $(0, 1)$, $(\pi/2, 0)$, and $(\pi, -1)$ are on the restricted cosine graph. Reversing the coordinates gives us three points on the graph of the inverse cosine function.



› **Figure 6**
Inverse cosine function.

››› EXPLORE-DISCUSS 2

A graphing calculator produced the graph in Figure 7 for $y_1 = \cos^{-1} x$, $-2 \le x \le 2$, and $0 \le y \le 4$. Explain why there are no parts of the graph on the intervals $[-2, -1)$ and $(1, 2]$.



› **Figure 7**

We complete the discussion by giving the cosine–inverse cosine identities:

› COSINE–INVERSE COSINE IDENTITIES

$$\cos(\cos^{-1} x) = x \qquad -1 \le x \le 1 \qquad f(f^{-1}(x)) = x$$
$$\cos^{-1}(\cos x) = x \qquad 0 \le x \le \pi \qquad f^{-1}(f(x)) = x$$

››› EXPLORE-DISCUSS 3

Evaluate each of the following with a calculator. Which illustrate a cosine–inverse cosine identity and which do not? Discuss why.

(A) $\cos(\cos^{-1} 0.2)$ **(B)** $\cos[\cos^{-1}(-2)]$

(C) $\cos^{-1}(\cos 2)$ **(D)** $\cos^{-1}[\cos(-3)]$

EXAMPLE 3 Exact Values

Find exact values without using a calculator.

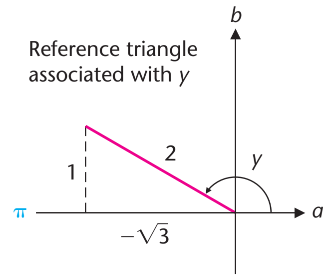
(A) $\arccos(-\sqrt{3}/2)$ (B) $\cos(\cos^{-1} 0.7)$ (C) $\sin[\cos^{-1}(-\frac{1}{3})]$

SOLUTIONS

(A) $y = \arccos(-\sqrt{3}/2)$ is equivalent to

$$\cos y = -\frac{\sqrt{3}}{2} \qquad 0 \le y \le \pi$$

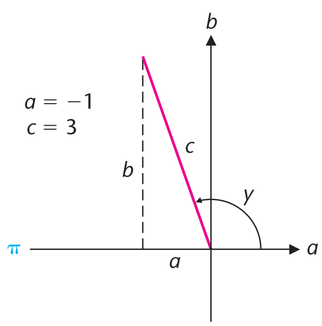
$$y = \frac{5\pi}{6} = \arccos\left(-\frac{\sqrt{3}}{2}\right)$$



[*Note:* $y \ne -5\pi/6$, even though $\cos(-5\pi/6) = -\sqrt{3}/2$ because y must be between 0 and π, inclusive.]

(B) $\cos(\cos^{-1} 0.7) = 0.7$ Cosine–inverse cosine identity, because $-1 \le 0.7 \le 1$

(C) Let $y = \cos^{-1}(-\frac{1}{3})$; then $\cos y = -\frac{1}{3}$, $0 \le y \le \pi$. Draw a reference triangle associated with y. Then $\sin y = \sin[\cos^{-1}(-\frac{1}{3})]$ can be determined directly from the triangle (after finding the third side) without actually finding y.



$$\begin{aligned} a^2 + b^2 &= c^2 \\ b &= \sqrt{3^2 - (-1^2)} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

Because $b > 0$ in quadrant II

Thus, $\sin[\cos^{-1}(-\frac{1}{3})] = \sin y = 2\sqrt{2}/3$.

MATCHED PROBLEM 3

Find exact values without using a calculator.

(A) $\arccos(\sqrt{2}/2)$ (B) $\cos^{-1}(\cos 3.05)$ (C) $\cot[\cos^{-1}(-1/\sqrt{5})]$

EXAMPLE 4

Calculator Values

Find to four significant digits using a calculator.

(A) $\arccos 0.4325$ (B) $\cos^{-1} 2.137$ (C) $\csc[\cos^{-1}(-0.0349)]$

SOLUTIONS

Set your calculator in radian mode.

(A) $\arccos 0.4325 = 1.124$

(B) $\cos^{-1} 2.137 = \text{Error}$ 2.137 is not in the domain of $\cos^{-1}$.

(C) $\csc[\cos^{-1}(-0.0349)] = 1.001$

MATCHED PROBLEM 4

Find to four significant digits using a calculator.

(A) $\cos^{-1}(0.6773)$ (B) $\arccos(-1.003)$ (C) $\cot[\cos^{-1}(-0.5036)]$

Inverse Tangent Function

To restrict the tangent function so that it becomes one-to-one, we choose the interval $(-\pi/2, \pi/2)$. Over this interval the restricted function passes the horizontal line test, and each range value is assumed exactly once as x moves across this restricted domain (Fig. 8). We use this restricted tangent function to define the *inverse tangent function.*

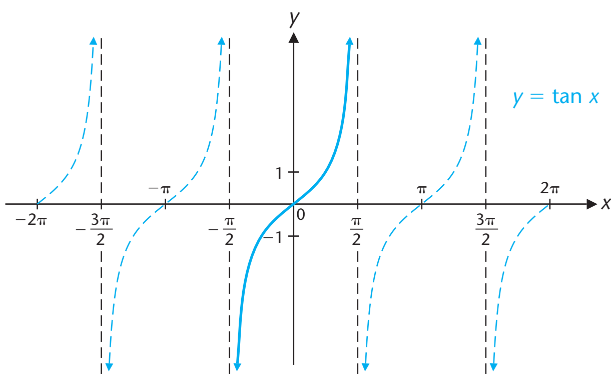


Figure 8 $y = \tan x$ is one-to-one over $(-\pi/2, \pi/2)$.

DEFINITION 3 Inverse Tangent Function

The **inverse tangent function,** denoted by $\tan^{-1}$ or arctan, is defined as the inverse of the restricted tangent function $y = \tan x$, $-\pi/2 < x < \pi/2$. Thus,

$$y = \tan^{-1} x \qquad \text{and} \qquad y = \arctan x$$

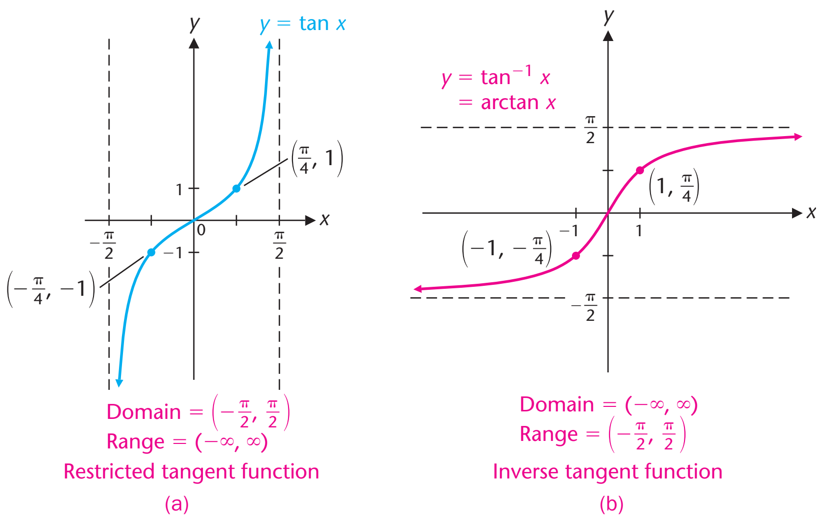
are equivalent to

$$\tan y = x \qquad \text{where} \qquad -\pi/2 < y < \pi/2 \text{ and } x \text{ is a real number}$$

In words, the inverse tangent of x, or the arctangent of x, is the number or angle y, $-\pi/2 < y < \pi/2$, whose tangent is x.

Figure 9 compares the graphs of the restricted tangent function and its inverse. Notice that $(-\pi/4, -1)$, $(0, 0)$, and $(\pi/4, 1)$ are on the restricted tangent graph. Reversing the coordinates gives us three points on the graph of the inverse tangent function. Also note that the vertical asymptotes become horizontal asymptotes for the inverse function.

Figure 9
Inverse tangent function.



We now state the tangent–inverse tangent identities.

TANGENT–INVERSE TANGENT IDENTITIES

$\tan(\tan^{-1} x) = x$	$-\infty < x < \infty$	$f(f^{-1}(x)) = x$
$\tan^{-1}(\tan x) = x$	$-\pi/2 < x < \pi/2$	$f^{-1}(f(x)) = x$

EXPLORE-DISCUSS 4

Evaluate each of the following with a calculator. Which illustrate a tangent–inverse tangent identity and which do not? Discuss why.

(A) $\tan(\tan^{-1} 30)$ **(B)** $\tan[\tan^{-1}(-455)]$

(C) $\tan^{-1}(\tan 1.4)$ **(D)** $\tan^{-1}[\tan(-3)]$

EXAMPLE 5 Exact Values

Find exact values without using a calculator.

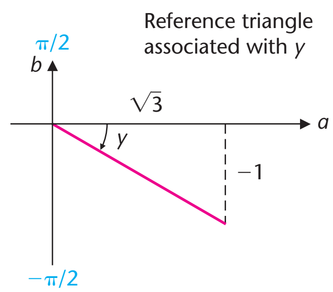
(A) $\tan^{-1}(-1/\sqrt{3})$ (B) $\tan^{-1}(\tan 0.63)$

SOLUTIONS

(A) $y = \tan^{-1}(-1/\sqrt{3})$ is equivalent to

$$\tan y = -\frac{1}{\sqrt{3}} \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$y = -\frac{\pi}{6} = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$



[*Note:* y cannot be $11\pi/6$ because y must be between $-\pi/2$ and $\pi/2$.]

(B) $\tan^{-1}(\tan 0.63) = 0.63$ Tangent–inverse tangent identity, because $-\pi/2 \le 0.63 \le \pi/2$

MATCHED PROBLEM 5

Find exact values without using a calculator.

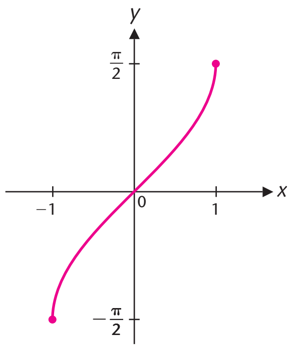
(A) $\arctan(-\sqrt{3})$ (B) $\tan(\tan^{-1} 43)$

Summary

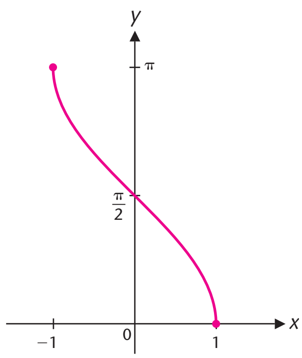
We summarize the definitions and graphs of the inverse trigonometric functions discussed so far for convenient reference.

SUMMARY OF $\sin^{-1}$, $\cos^{-1}$, AND $\tan^{-1}$

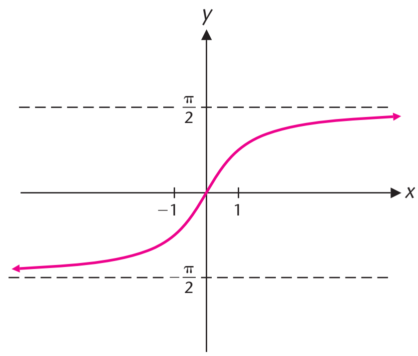
$y = \sin^{-1} x$	is equivalent to	$x = \sin y$	where $-1 \le x \le 1$, $-\pi/2 \le y \le \pi/2$
$y = \cos^{-1} x$	is equivalent to	$x = \cos y$	where $-1 \le x \le 1$, $0 \le y \le \pi$
$y = \tan^{-1} x$	is equivalent to	$x = \tan y$	where $-\infty < x < \infty$, $-\pi/2 < y < \pi/2$



$y = \sin^{-1} x$
Domain $= [-1, 1]$
Range $= [-\frac{\pi}{2}, \frac{\pi}{2}]$



$y = \cos^{-1} x$
Domain $= [-1, 1]$
Range $= [0, \pi]$



$y = \tan^{-1} x$
Domain $= (-\infty, \infty)$
Range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Inverse Cotangent, Secant, and Cosecant Functions (Optional)

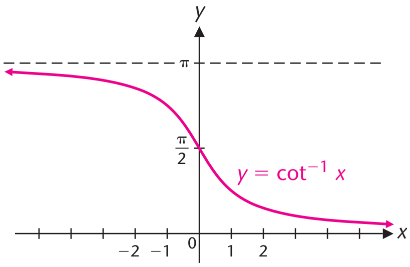
For completeness, we include the definitions and graphs of the inverse cotangent, secant, and cosecant functions.

› DEFINITION 4 Inverse Cotangent, Secant, and Cosecant Functions

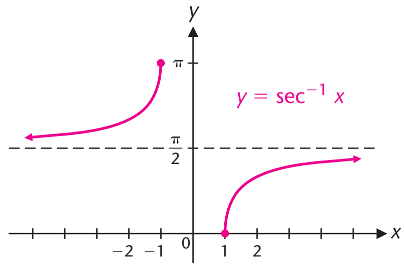
$y = \cot^{-1} x$ is equivalent to $x = \cot y$ where $0 < y < \pi$, $-\infty < x < \infty$

$y = \sec^{-1} x$ is equivalent to $x = \sec y$ where $0 \le y \le \pi$, $y \ne \pi/2$, $|x| \ge 1$

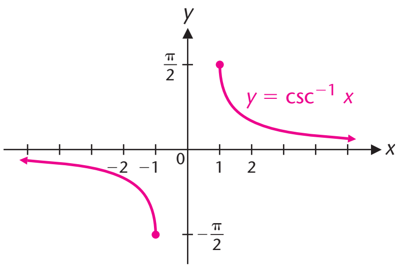
$y = \csc^{-1} x$ is equivalent to $x = \csc y$ where $-\pi/2 \le y \le \pi/2$, $y \ne 0$, $|x| \ge 1$



Domain: All real numbers
Range: $0 < y < \pi$



Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi$, $y \ne \pi/2$



Domain: $x \le -1$ or $x \ge 1$
Range: $-\pi/2 \le y \le \pi/2$, $y \ne 0$

[*Note:* The definitions of $\sec^{-1}$ and $\csc^{-1}$ are not universally agreed upon.]

ANSWERS TO MATCHED PROBLEMS

1. (A) $\pi/4$ (B) -0.4 (C) $-1/2$
2. (A) 0.2945 (B) Not defined (C) -2.724
3. (A) $\pi/4$ (B) 3.05 (C) $-1/2$
4. (A) 0.8267 (B) Not defined (C) -0.5829
5. (A) $-\pi/3$ (B) 43

6-6 Exercises

Unless stated to the contrary, the inverse trigonometric functions are assumed to have real number ranges (use radian mode in calculator problems). A few problems involve ranges with angles in degree measure, and these are clearly indicated (use degree mode in calculator problems).

In Problems 1–12, find exact values without using a calculator.

1. $\cos^{-1} 0$

2. $\sin^{-1} 0$

3. $\arcsin(\sqrt{3}/2)$

4. $\arccos(\sqrt{3}/2)$

5. $\arctan\sqrt{3}$

6. $\tan^{-1} 1$

7. $\sin^{-1}(\sqrt{2}/2)$

8. $\cos^{-1}(\frac{1}{2})$

9. $\arccos 1$

10. $\arctan(1/\sqrt{3})$

11. $\sin^{-1}(\frac{1}{2})$

12. $\tan^{-1} 0$

In Problems 13–18, evaluate to four significant digits using a calculator.

13. $\sin^{-1} 0.9103$

14. $\cos^{-1} 0.4038$

15. $\arctan 103.7$

16. $\tan^{-1} 43.09$

17. $\arccos 3.051$

18. $\arcsin 1.131$

In Problems 19–34, find exact values without using a calculator.

19. $\arcsin(-\sqrt{2}/2)$

20. $\arccos(-\frac{1}{2})$

21. $\tan^{-1}(-\sqrt{3})$

22. $\tan^{-1}(-1)$

23. $\tan(\tan^{-1} 25)$

24. $\sin[\sin^{-1}(-0.6)]$

25. $\cos^{-1}(\cos 2.3)$

26. $\tan^{-1}[\tan(-1.5)]$

27. $\sin(\cos^{-1}\sqrt{3}/2)$

28. $\tan[\cos^{-1}(\frac{1}{2})]$

29. $\csc[\tan^{-1}(-1)]$

30. $\cos[\sin^{-1}(-\sqrt{2}/2)]$

31. $\sin^{-1}[\sin\pi]$

32. $\cos^{-1}[\cos(-\pi/2)]$

33. $\cos^{-1}[\cos(4\pi/3)]$

34. $\sin^{-1}[\sin(5\pi/4)]$

In Problems 35–40, evaluate to four significant digits using a calculator.

35. $\arctan(-10.04)$

36. $\tan^{-1}(-4.038)$

37. $\cot[\cos^{-1}(-0.7003)]$

38. $\sec[\sin^{-1}(-0.0399)]$

39. $\sqrt{5 + \cos^{-1}(1 - \sqrt{2})}$

40. $\sqrt{2} + \tan^{-1}\sqrt[3]{5}$

In Problems 41–46, find the exact degree measure of each without the use of a calculator.

41. $\sin^{-1}(-\sqrt{2}/2)$

42. $\cos^{-1}(-\frac{1}{2})$

43. $\arctan(-\sqrt{3})$

44. $\arctan(-1)$

45. $\cos^{-1}(-1)$

46. $\sin^{-1}(-1)$

In Problems 47–52, find the degree measure of each to two decimal places using a calculator set in degree mode.

47. $\cos^{-1} 0.7253$

48. $\tan^{-1} 12.4304$

49. $\arcsin(-0.3662)$

50. $\arccos(-0.9206)$

51. $\tan^{-1}(-837)$

52. $\sin^{-1}(-0.7071)$

53. Evaluate $\sin^{-1}(\sin 2)$ with a calculator set in radian mode, and explain why this does or does not illustrate the inverse sine–sine identity.

54. Evaluate $\cos^{-1}[\cos(-0.5)]$ with a calculator set in radian mode, and explain why this does or does not illustrate the inverse cosine–cosine identity.

In Problems 55–62, determine whether the statement is true or false. Explain.

55. None of the six trigonometric functions is one-to-one.

56. Each of the six inverse trigonometric functions is one-to-one.

57. Each of the six inverse trigonometric functions is periodic.

58. Each of the six inverse trigonometric functions is bounded.

59. The function $y = \sin^{-1} x$ is odd.

60. The function $y = \cos^{-1} x$ is even.

61. None of the six inverse trigonometric functions has a turning point.

62. None of the six inverse trigonometric functions has a zero.

In Problems 63–70, graph each function over the indicated interval.

63. $y = \sin^{-1} x, -1 \le x \le 1$

64. $y = \cos^{-1} x, -1 \le x \le 1$

65. $y = \cos^{-1}(x/3), -3 \le x \le 3$

66. $y = \sin^{-1}(x/2),\ -2 \le x \le 2$

67. $y = \sin^{-1}(x - 2),\ 1 \le x \le 3$

68. $y = \cos^{-1}(x + 1),\ -2 \le x \le 0$

69. $y = \tan^{-1}(2x - 4),\ -2 \le x \le 6$

70. $y = \tan^{-1}(2x + 3),\ -5 \le x \le 2$

71. The identity $\cos(\cos^{-1} x) = x$ is valid for $-1 \le x \le 1$.
(A) Graph $y = \cos(\cos^{-1} x)$ for $-1 \le x \le 1$.
(B) What happens if you graph $y = \cos(\cos^{-1} x)$ over a larger interval, say $-2 \le x \le 2$? Explain.

72. The identity $\sin(\sin^{-1} x) = x$ is valid for $-1 \le x \le 1$.
(A) Graph $y = \sin(\sin^{-1} x)$ for $-1 \le x \le 1$.
(B) What happens if you graph $y = \sin(\sin^{-1} x)$ over a larger interval, say $-2 \le x \le 2$? Explain.

In Problems 73–76, write each expression as an algebraic expression in x free of trigonometric or inverse trigonometric functions.

73. $\cos(\sin^{-1} x)$ **74.** $\sin(\cos^{-1} x)$

75. $\cos(\arctan x)$ **76.** $\tan(\arcsin x)$

In Problems 77 and 78, find $f^{-1}(x)$. How must x be restricted in $f^{-1}(x)$?

77. $f(x) = 4 + 2\cos(x - 3),\ 3 \le x \le (3 + \pi)$

78. $f(x) = 3 + 5\sin(x - 1),\ (1 - \pi/2) \le x \le (1 + \pi/2)$

79. The identity $\cos^{-1}(\cos x) = x$ is valid for $0 \le x \le \pi$.
(A) Graph $y = \cos^{-1}(\cos x)$ for $0 \le x \le \pi$.
(B) What happens if you graph $y = \cos^{-1}(\cos x)$ over a larger interval, say $-2\pi \le x \le 2\pi$? Explain.

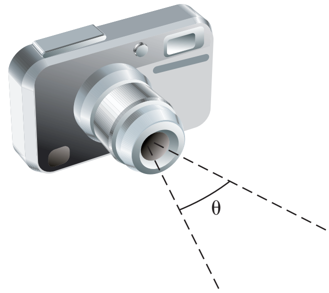
80. The identity $\sin^{-1}(\sin x) = x$ is valid for $-\pi/2 \le x \le \pi/2$.
(A) Graph $y = \sin^{-1}(\sin x)$ for $-\pi/2 \le x \le \pi/2$.
(B) What happens if you graph $y = \sin^{-1}(\sin x)$ over a larger interval, say $-2\pi \le x \le 2\pi$? Explain.

APPLICATIONS

81. PHOTOGRAPHY The viewing angle changes with the focal length of a camera lens. A 28-millimeter wide-angle lens has a wide viewing angle and a 300-millimeter telephoto lens has a narrow viewing angle. For a 35-millimeter format camera the viewing angle θ, in degrees, is given by

$$\theta = 2\tan^{-1}\frac{21.634}{x}$$

where x is the focal length of the lens being used. What is the viewing angle (in decimal degrees to two decimal places) of a 28-millimeter lens? Of a 100-millimeter lens?



82. PHOTOGRAPHY Referring to Problem 81, what is the viewing angle (in decimal degrees to two decimal places) of a 17-millimeter lens? Of a 70-millimeter lens?

83. (A) Graph the function in Problem 81 in a graphing calculator using degree mode. The graph should cover lenses with focal lengths from 10 millimeters to 100 millimeters.
(B) What focal-length lens, to two decimal places, would have a viewing angle of 40°? Solve by graphing $\theta = 40$ and $\theta = 2\tan^{-1}(21.634/x)$ in the same viewing window and finding the point of intersection using an approximation routine.

84. (A) Graph the function in Problem 81 in a graphing calculator, in degree mode, with the graph covering lenses with focal lengths from 100 millimeters to 1,000 millimeters.
(B) What focal length lens, to two decimal places, would have a viewing angle of 10°? Solve by graphing $\theta = 10$ and $\theta = \tan^{-1}(21.634/x)$ in the same viewing window and finding the point of intersection using an approximation routine.

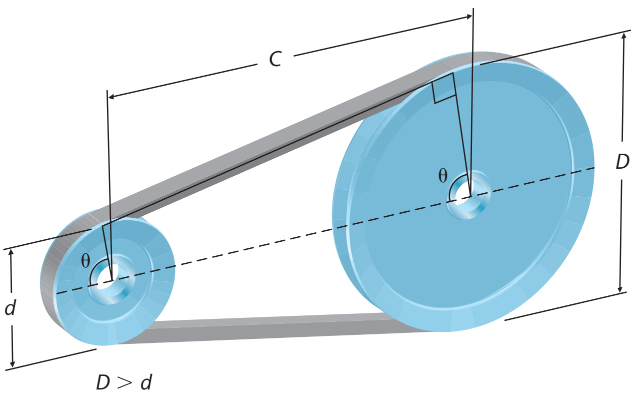
★**85.** ENGINEERING The length of the belt around the two pulleys in the figure is given by

$$L = \pi D + (d - D)\theta + 2C\sin\theta$$

where θ (in radians) is given by

$$\theta = \cos^{-1}\frac{D - d}{2C}$$

Verify these formulas, and find the length of the belt to two decimal places if $D = 4$ inches, $d = 2$ inches, and $C = 6$ inches.



★**86. ENGINEERING** For Problem 85, find the length of the belt if $D = 6$ inches, $d = 4$ inches, and $C = 10$ inches.

87. ENGINEERING The function

$$y_1 = 4\pi - 2\cos^{-1}\frac{1}{x} + 2x\sin\left(\cos^{-1}\frac{1}{x}\right)$$

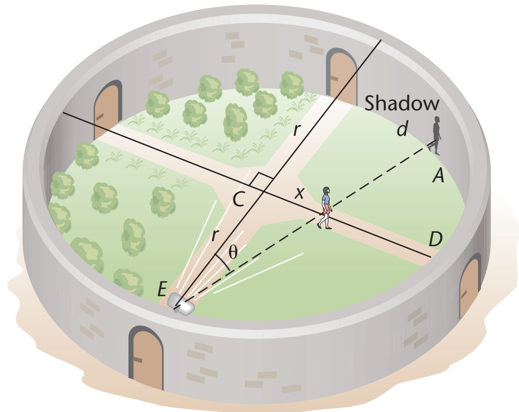
represents the length of the belt around the two pulleys in Problem 85 when the centers of the pulleys are x inches apart.
(A) Graph y_1 in a graphing calculator (in radian mode), with the graph covering pulleys with their centers from 3 to 10 inches apart.
(B) How far, to two decimal places, should the centers of the two pulleys be placed to use a belt 24 inches long? Solve by graphing y_1 and $y_2 = 24$ in the same viewing window and finding the point of intersection using an approximation routine.

88. ENGINEERING The function

$$y_1 = 6\pi - 2\cos^{-1}\frac{1}{x} + 2x\sin\left(\cos^{-1}\frac{1}{x}\right)$$

represents the length of the belt around the two pulleys in Problem 86 when the centers of the pulleys are x inches apart.
(A) Graph y_1 in a graphing calculator (in radian mode), with the graph covering pulleys with their centers from 3 to 20 inches apart.
(B) How far, to two decimal places, should the centers of the two pulleys be placed to use a belt 36 inches long? Solve by graphing y_1 and $y_2 = 36$ in the same viewing window and finding the point of intersection using an approximation routine.

★**89. MOTION** The figure represents a circular courtyard surrounded by a high stone wall. A floodlight located at E shines into the courtyard.



(A) If a person walks x feet away from the center along DC, show that the person's shadow will move a distance given by

$$d = 2r\theta = 2r\tan^{-1}\frac{x}{r}$$

where θ is in radians. [*Hint:* Draw a line from A to C.]

(B) Find d to two decimal places if $r = 100$ feet and $x = 40$ feet.

★**90. MOTION** In Problem 89, find d for $r = 50$ feet and $x = 25$ feet.

CHAPTER 6 Review

6-1 Angles and Their Measure

An **angle** is formed by rotating (in a plane) a ray m, called the **initial side** of the angle, around its endpoint until it coincides with a ray n, called the **terminal side** of the angle. The common endpoint of m and n is called the **vertex.** If the rotation is counterclockwise, the angle is **positive;** if clockwise, **negative.** Two angles are **coterminal** if they have the same initial and terminal sides.

An angle is in **standard position** in a rectangular coordinate system if its vertex is at the origin and its initial side is along the positive x axis. **Quadrantal angles** have their terminal sides on a coordinate axis. An angle of **1 degree** is $\frac{1}{360}$ of a complete rotation. Two positive angles are **complementary** if their sum is 90°; they are **supplementary** if their sum is 180°.

An angle of **1 radian** is a central angle of a circle subtended by an arc having the same length as the radius.

$$\text{Radian measure: } \theta = \frac{s}{r}$$

$$\text{Radian–degree conversion: } \frac{\theta_{\text{deg}}}{180°} = \frac{\theta_{\text{rad}}}{\pi \text{ radians}}$$

If a point P moves through an angle θ and arc length s, in time t, on the circumference of a circle of radius r, then the (average) **linear speed** of P is

$$v = \frac{s}{t}$$

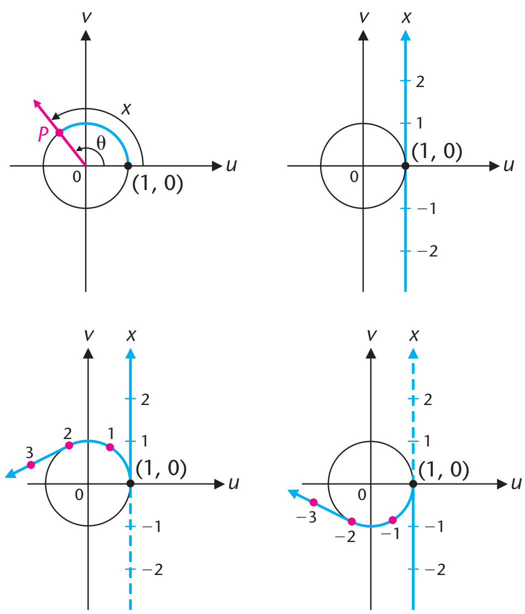
and the (average) **angular speed** is

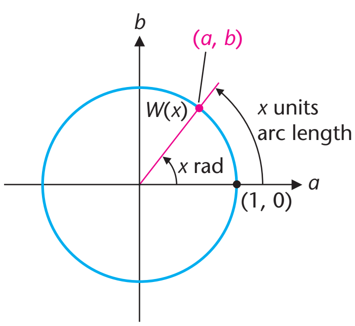
$$\omega = \frac{\theta}{t}$$

Because $s = r\theta$ it follows that $v = r\omega$.

6-2 Trigonometric Functions: A Unit Circle Approach

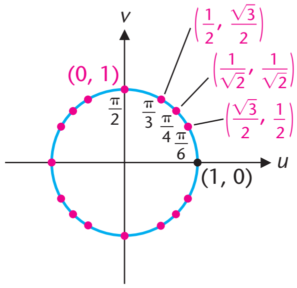
If θ is a positive angle in standard position, and P is the point of intersection of the terminal side of θ with the unit circle, then the radian measure of θ equals the length x of the arc opposite θ; and if θ is negative, the radian measure of θ equals the negative of the length of the intercepted arc. The function W that associates with each real number x the point $W(x) = P$ is called the **wrapping function,** and the point P is called a **circular point.** The function $W(x)$ can be visualized as a wrapping of the real number line, with origin at (1, 0), around the unit circle—the positive real axis is wrapped counterclockwise and the negative real axis is wrapped clockwise—so that each real number is paired with a unique circular point. The function $W(x)$ is not one-to-one: for example, each of the real numbers $2\pi k$, k any integer, corresponds to the circular point (1, 0).





The coordinates of key circular points in the first quadrant can be found using simple geometric facts; the coordinates of the circular point associated with any multiple of $\pi/6$ or $\pi/4$ can then be determined using symmetry properties.

Coordinates of Key Circular Points



The six **trigonometric functions—sine, cosine, tangent, cotangent, secant, and cosecant**—are defined in terms of the coordinates (a, b) of the circular point $W(x)$ that lies on the terminal side of the angle with radian measure x:

$$\sin x = b \qquad \csc x = \frac{1}{b} \quad b \neq 0$$

$$\cos x = a \qquad \sec x = \frac{1}{a} \quad a \neq 0$$

$$\tan x = \frac{b}{a} \quad a \neq 0 \qquad \cot x = \frac{a}{b} \quad b \neq 0$$

The trigonometric functions of any multiple of $\pi/6$ or $\pi/4$ can be determined exactly from the coordinates of the circular point. A graphing calculator can be used to graph the trigonometric functions and approximate their values at arbitrary angles.

6-3 Solving Right Triangles

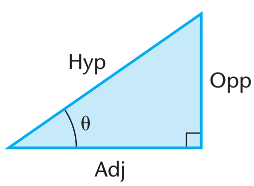
A **right triangle** is a triangle with one 90° angle. To **solve a right triangle** is to find all unknown angles and sides, given the measures of two sides or the measures of one side and an acute angle.

Trigonometric Ratios

$$\sin\theta = \frac{\text{Opp}}{\text{Hyp}} \qquad \csc\theta = \frac{\text{Hyp}}{\text{Opp}}$$

$$\cos\theta = \frac{\text{Adj}}{\text{Hyp}} \qquad \sec\theta = \frac{\text{Hyp}}{\text{Adj}}$$

$$\tan\theta = \frac{\text{Opp}}{\text{Adj}} \qquad \cot\theta = \frac{\text{Adj}}{\text{Opp}}$$



Computational Accuracy

Angle to Nearest	Significant Digits for Side Measure
1°	2
10′ or 0.1°	3
1′ or 0.01°	4
10″ or 0.001°	5

6-4 Properties of Trigonometric Functions

The definition of the trigonometric functions implies that the following **basic identities** hold true for all replacements of x by real numbers for which both sides of an equation are defined:

Reciprocal identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

Quotient identities

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Identities for negatives

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

Pythagorean identity

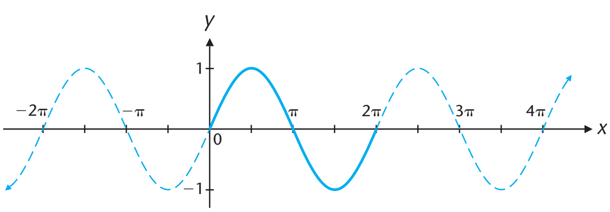
$$\sin^2 x + \cos^2 x = 1$$

A function f is **periodic** if there exists a positive real number p such that

$$f(x + p) = f(x)$$

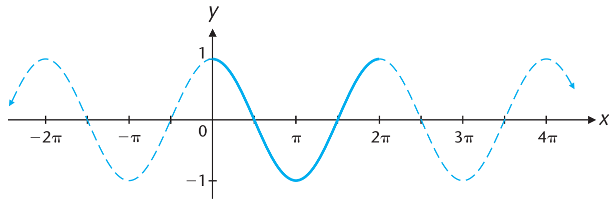
for all x in the domain of f. The smallest such positive p, if it exists, is called the **fundamental period of *f***, or often just the **period of *f***. All the trigonometric functions are periodic.

Graph of $y = \sin x$:



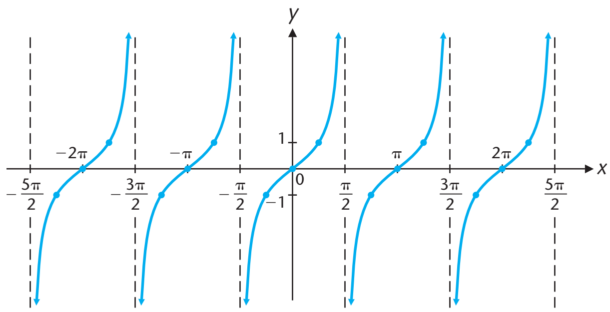
Period: 2π
Domain: All real numbers
Range: $[-1, 1]$

Graph of $y = \cos x$:



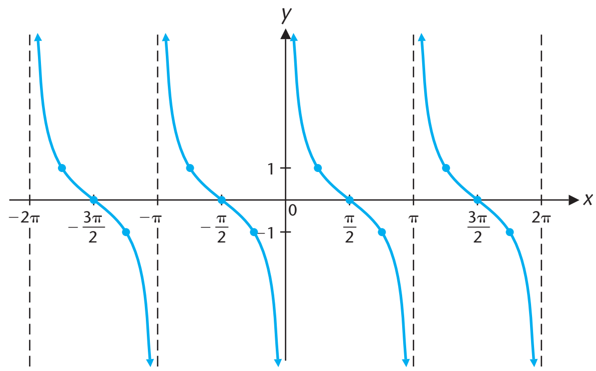
Period: 2π
Domain: All real numbers
Range: $[-1, 1]$

Graph of $y = \tan x$:



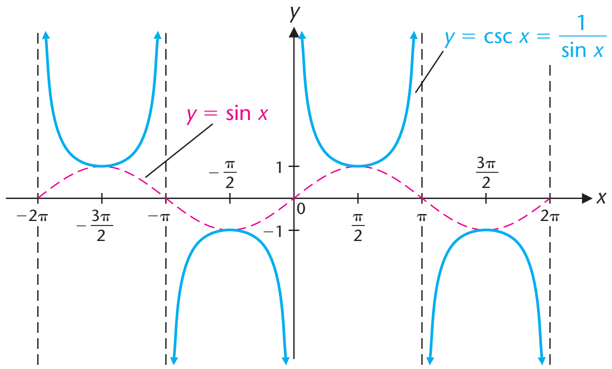
Period: π
Domain: All real numbers except $\pi/2 + k\pi$, k an integer
Range: All real numbers

Graph of $y = \cot x$:



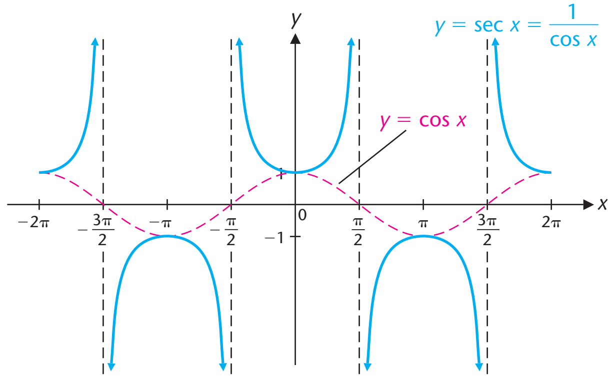
Period: π
Domain: All real numbers except $k\pi$, k an integer
Range: All real numbers

Graph of $y = \csc x$:



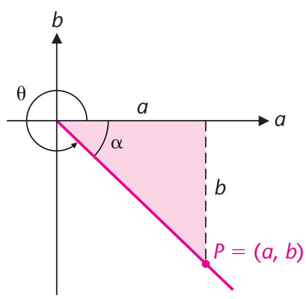
Period: 2π
Domain: All real numbers except $k\pi$, k an integer
Range: All real numbers y such that $y \le -1$ or $y \ge 1$

Graph of $y = \sec x$:



Period: 2π
Domain: All real numbers except $\pi/2 + k\pi$, k an integer
Range: All real numbers y such that $y \le -1$ or $y \ge 1$

Associated with each angle that does not terminate on a coordinate axis is a **reference triangle** for θ. The reference triangle is formed by drawing a perpendicular from point $P = (a, b)$ on the terminal side of θ to the horizontal axis. The **reference angle** α is the acute angle, always taken positive, between the terminal side of θ and the horizontal axis as indicated in the following figure.



Reference Triangle
$(a, b) \ne (0, 0)$
α is always positive

6-5 More General Trigonometric Functions and Models

Let A, B, C be constants such that $A \ne 0$ and $B > 0$.
If $y = A \sin(Bx + C)$ or $y = A \cos(Bx + C)$:

$$\text{Amplitude} = |A| \qquad \text{Period} = \frac{2\pi}{B} \qquad \text{Phase shift} = \frac{-C}{B}$$

If $y = A \sec(Bx + C)$ or $y = \csc(Bx + C)$:

$$\text{Period} = \frac{2\pi}{B} \qquad \text{Phase shift} = \frac{-C}{B}$$

If $y = A \tan(Bx + C)$ or $y = A \cot(Bx + C)$:

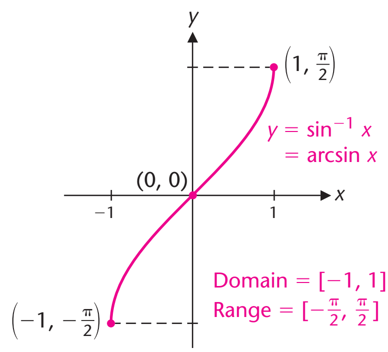
$$\text{Period} = \frac{\pi}{B} \qquad \text{Phase shift} = \frac{-C}{B}$$

(Amplitude is not defined for the secant, cosecant, tangent, and cotangent functions, all of which are unbounded.)

Sinusoidal regression is used to find the function of the form $y = A \sin(Bx + C) + k$ that best fits a set of data points.

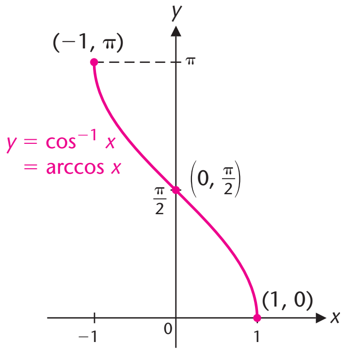
6-6 Inverse Trigonometric Functions

$y = \sin^{-1} x = \arcsin x$ if and only if $\sin y = x$, $-\pi/2 \le y \le \pi/2$ and $-1 \le x \le 1$.



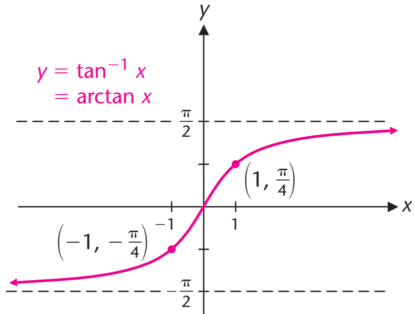
Inverse sine function

$y = \cos^{-1} x = \arccos x$ if and only if $\cos y = x$, $0 \le y \le \pi$ and $-1 \le x \le 1$.



Domain = [−1, 1]
Range = [0, π]
Inverse cosine function

$y = \tan^{-1} x = \arctan x$ if and only if $\tan y = x$, $-\pi/2 < y < \pi/2$ and x is any real number.



Domain = $(-\infty, \infty)$
Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
Inverse tangent function

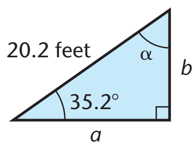
CHAPTER 6 Review Exercises

Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

1. Find the radian measure of a central angle opposite an arc 15 centimeters long on a circle of radius 6 centimeters.

2. In a circle of radius 3 centimeters, find the length of an arc opposite an angle of 2.5 radians.

3. Solve the triangle:



4. Find the reference angle associated with each angle θ.
(A) $\theta = \pi/3$ (B) $\theta = -120°$
(C) $\theta = -13\pi/6$ (D) $\theta = 210°$

5. In which quadrants is each negative?
(A) $\sin \theta$ (B) $\cos \theta$ (C) $\tan \theta$

6. If $(4, -3)$ is on the terminal side of angle θ, find
(A) $\sin \theta$ (B) $\sec \theta$ (C) $\cot \theta$

7. Complete Table 1 using exact values. Do not use a calculator.

Table 1

θ°	θ rad	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
0°					ND*		
30°							
45°	$\pi/4$		$1/\sqrt{2}$				
60°							
90°							
180°							
270°							
360°							

*ND = Not defined

8. What is the period of each of the following?
(A) $y = \cos x$ (B) $y = \csc x$ (C) $y = \tan x$

9. Indicate the domain and range of each.
(A) $y = \sin x$ (B) $y = \tan x$

10. Sketch a graph of $y = \sin x$, $-2\pi \le x \le 2\pi$.

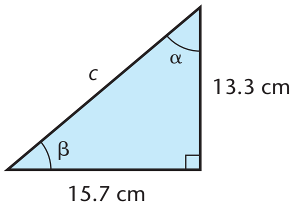
11. Sketch a graph of $y = \cot x$, $-\pi < x < \pi$.

12. Verbally describe the meaning of a central angle in a circle with radian measure 0.5.

13. Describe the smallest shift of the graph of $y = \sin x$ that produces the graph of $y = \cos x$.

14. Change 1.37 radians to decimal degrees to two decimal places.

15. Solve the triangle:



16. Indicate whether the angle is a quadrant I, II, III, or IV angle or a quadrantal angle.
(A) $-210°$ (B) $5\pi/2$ (C) 4.2 radians

17. Which of the following angles are coterminal with 120°?
(A) $-240°$ (B) $-7\pi/6$ (C) 840°

18. Which of the following have the same value as cos 3?
(A) cos 3° (B) cos (3 radians) (C) $\cos(3 + 2\pi)$

19. For which values of x, $0 \le x \le 2\pi$, is each of the following not defined?
(A) $\tan x$ (B) $\cot x$ (C) $\csc x$

20. A circular point $P = (a, b)$ moves clockwise around the circumference of a unit circle starting at (1, 0) and stops after covering a distance of 8.305 units. Explain how you would find the coordinates of point P at its final position and how you would determine which quadrant P is in. Find the coordinates of P to three decimal places and the quadrant for the final position of P.

In Problems 21–36, evaluate exactly without the use of a calculator.

21. $\tan 0$

22. $\sec 90°$

23. $\cos^{-1} 1$

24. $\cos\left(-\frac{3\pi}{4}\right)$

25. $\sin^{-1} \frac{\sqrt{2}}{2}$

26. $\csc 300°$

27. $\arctan \sqrt{3}$

28. $\sin 570°$

29. $\tan^{-1}(-1)$

30. $\cot\left(-\frac{4\pi}{3}\right)$

31. $\arcsin\left(-\frac{1}{2}\right)$

32. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

33. $\cos(\cos^{-1} 0.33)$

34. $\csc[\tan^{-1}(-1)]$

35. $\sin\left[\arccos\left(-\frac{1}{2}\right)\right]$

36. $\tan\left(\sin^{-1}\frac{-4}{5}\right)$

Evaluate Problems 37–44 to four significant digits using a calculator.

37. $\cos 423.7°$

38. $\tan 93°46'17''$

39. $\sec(-2.073)$

40. $\sin^{-1}(-0.8277)$

41. $\arccos(-1.3281)$

42. $\tan^{-1} 75.14$

43. $\csc[\cos^{-1}(-0.4081)]$

44. $\sin^{-1}(\tan 1.345)$

45. Find the exact degree measure of each without a calculator.
(A) $\theta = \sin^{-1}(-\frac{1}{2})$ (B) $\theta = \arccos(-\frac{1}{2})$

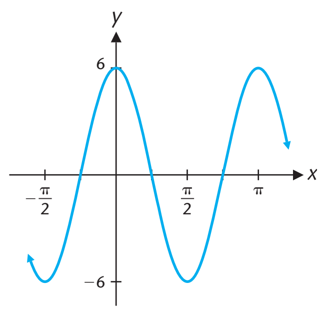
46. Find the degree measure of each to two decimal places using a calculator.
(A) $\theta = \cos^{-1}(-0.8763)$ (B) $\theta = \arctan 7.3771$

47. Evaluate $\cos^{-1}[\cos(-2)]$ with a calculator set in radian mode, and explain why this does or does not illustrate the inverse cosine–cosine identity.

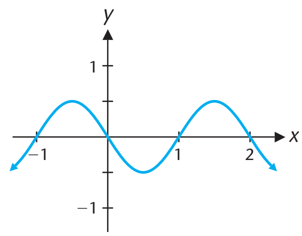
48. Sketch a graph of $y = -2\cos \pi x$, $-1 \le x \le 3$. Indicate amplitude A and period P.

49. Sketch a graph of $y = -2 + 3\sin(x/2)$, $-4\pi \le x \le 4\pi$.

50. Find the equation of the form $y = A\cos Bx$ that has the graph shown here.



51. Find the equation of the form $y = A\sin Bx$ that has the graph shown here.



52. Describe the smallest shift and/or reflection that transforms the graph of $y = \tan x$ into the graph of $y = \cot x$.

53. Simplify each of the following using appropriate basic identities:
(A) $\sin(-x)\cot(-x)$ (B) $\frac{\sin^2 x}{1 - \sin^2 x}$

54. Sketch a graph of $y = 3 \sin [(x/2) + (\pi/2)]$ over the interval $-4\pi \le x \le 4\pi$.

55. Indicate the amplitude A, period P, and phase shift for the graph of $y = -2 \cos [(\pi/2)x - (\pi/4)]$. Do not graph.

56. Sketch a graph of $y = \cos^{-1} x$, and indicate the domain and range.

57. Graph $y = 1/(1 + \tan^2 x)$ in a graphing calculator that displays at least two full periods of the graph. Find an equation of the form $y = k + A \sin Bx$ or $y = k + A \cos Bx$ that has the same graph.

58. Graph each equation in a graphing calculator and find an equation of the form $y = A \tan Bx$ or $y = A \cot Bx$ that has the same graph as the given equation. Select the dimensions of the viewing window so that at least two periods are visible.

(A) $y = \dfrac{2 \sin^2 x}{\sin 2x}$ (B) $y = \dfrac{2 \cos^2 x}{\sin 2x}$

59. Determine whether each function is even, odd, or neither.

(A) $f(x) = \dfrac{1}{1 + \tan^2 x}$

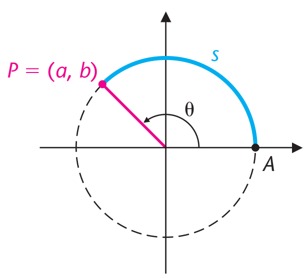
(B) $g(x) = \dfrac{1}{1 + \tan x}$

In Problems 60 and 61, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

60. If α and β are the acute angles of a right triangle, then $\sin \alpha = \csc \beta$.

61. If α and β are the acute angles of a right triangle and $\alpha = \beta$, then all six trigonometric functions of α are greater than $\frac{1}{2}$ and less than $\frac{3}{2}$.

62. If in the figure the coordinates of A are $(8, 0)$ and arc length s is 20 units, find:
(A) The exact radian measure of θ
(B) The coordinates of P to three significant digits



63. Find exactly the least positive real number for which
(A) $\cos x = -\frac{1}{2}$ (B) $\csc x = -\sqrt{2}$

64. Sketch a graph of $y = \sec x$, $-\pi/2 < x < 3\pi/2$.

65. Sketch a graph of $y = \tan^{-1} x$, and indicate the domain and range.

66. Indicate the period P and phase shift for the graph of $y = -5 \tan (\pi x + \pi/2)$. Do not graph.

67. Indicate the period and phase shift for the graph of $y = 3 \csc (x/2 - \pi/4)$. Do not graph.

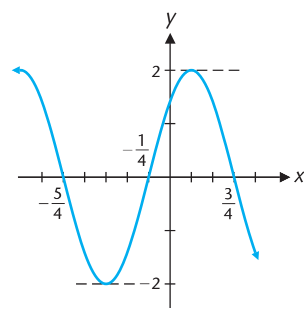
68. Indicate whether each is symmetrical with respect to the x axis, y axis, or origin.
(A) Sine (B) Cosine (C) Tangent

69. Write as an algebraic expression in x free of trigonometric or inverse trigonometric functions:

$$\sec (\sin^{-1} x)$$

70. Try to calculate each of the following on your calculator. Explain the results.
(A) $\csc (-\pi)$ (B) $\tan (-3\pi/2)$ (C) $\sin^{-1} 2$

71. The accompanying graph is a graph of an equation of the form $y = A \sin (Bx + C)$, $-1 < -C/B < 0$. Find the equation.



72. Graph $y = 1.2 \sin 2x + 1.6 \cos 2x$ in a graphing calculator. (Select the dimensions of the viewing window so that at least two periods are visible.) Find an equation of the form $y = A \sin (Bx + C)$ that has the same graph as the given equation. Find A and B exactly and C to three decimal places. Use the x intercept closest to the origin as the phase shift.

73. A particular waveform is approximated by the first six terms of a Fourier series:

$$y = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \frac{\sin 9x}{9} + \frac{\sin 11x}{11}\right)$$

(A) Graph this equation in a graphing calculator for $-3\pi \le x \le 3\pi$ and $-2 \le y \le 2$.
(B) The graph in part A approximates a waveform that is made up entirely of straight line segments. Sketch by hand the waveform that the Fourier series approximates.

This waveform is called a **pulse wave** or a **square wave,** and is used, for example, to test distortion and to synchronize operations in computers.

APPLICATIONS

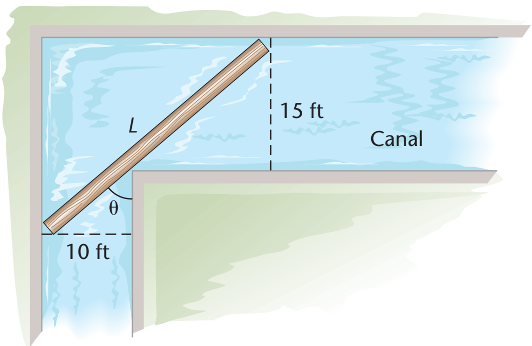
74. ASTRONOMY A line from the sun to the Earth sweeps out an angle of how many radians in 73 days? Express the answer in terms of π.

★**75. GEOMETRY** Find the perimeter of a square inscribed in a circle of radius 5.00 centimeters.

76. ANGULAR SPEED A wind turbine of rotor diameter 40 feet makes 80 revolutions per minute. Find the angular speed (in radians per second) and the linear speed (in feet per second) of the rotor tip.

★**77. ALTERNATING CURRENT** The current I in alternating electrical current has an amplitude of 30 amperes and a period of $\frac{1}{60}$ second. If $I = 30$ amperes when $t = 0$, find an equation of the form $I = A \cos Bt$ that gives the current at any time $t \geq 0$.

78. RESTRICTED ACCESS A 10-foot-wide canal makes a right turn into a 15-foot-wide canal. Long narrow logs are to be floated through the canal around the right angle turn (see the figure). We are interested in finding the longest log that will go around the corner, ignoring the log's diameter.



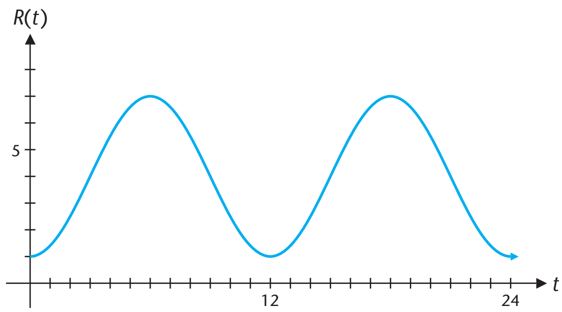
(A) Express the length L of the line that touches the two outer sides of the canal and the inside corner in terms of θ.
(B) Complete Table 2, each to one decimal place, and estimate from the table the longest log to the nearest foot that can make it around the corner. (The longest log is the shortest distance L.)

Table 2

θ (radians)	0.4	0.5	0.6	0.7	0.8	0.9	1.0
L (feet)	42.0						

(C) Graph the function in part A in a graphing calculator and use an approximation method to find the shortest distance L to one decimal place; hence, the length of the longest log that can make it around the corner.
(D) Explain what happens to the length L as θ approaches 0 or $\pi/2$.

79. MODELING SEASONAL BUSINESS CYCLES A soft drink company has revenues from sales over a 2-year period as shown by the accompanying graph, where $R(t)$ is revenue (in millions of dollars) for a month of sales t months after February 1.
(A) Find an equation of the form $R(t) = k + A \cos Bt$ that produces this graph, and check the result by graphing.
(B) Verbally interpret the graph



80. MODELING TEMPERATURE VARIATION The 30-year average monthly temperature, °F, for each month of the year for Los Angeles is given in Table 3 (*World Almanac*).
(A) Using 1 month as the basic unit of time, enter the data for a 2-year period in your graphing calculator and produce a scatter plot in the viewing window. Choose $40 \leq y \leq 90$ for the viewing window.
(B) It appears that a sine curve of the form

$$y = k + A \sin (Bx + C)$$

will closely model these data. The constants k, A, and B are easily determined from Table 3. To estimate C, visually estimate to one decimal place the smallest positive phase shift from the plot in part A. After determining A, B, k, and C, write the resulting equation. (Your value of C may differ slightly from the answer at the back of the book.)
(C) Plot the results of parts A and B in the same viewing window. (An improved fit may result by adjusting your value of C slightly.)
(D) If your graphing calculator has a sinusoidal regression feature, check your results from parts B and C by finding and plotting the regression equation.

Table 3

x (months)	1	2	3	4	5	6	7	8	9	10	11	12
y (temperature)	58	60	61	63	66	70	74	75	74	70	63	58

CHAPTER 6

››› GROUP ACTIVITY A Predator–Prey Analysis Involving Mountain Lions and Deer

In some western state wilderness areas, deer and mountain lion populations are interrelated, because the mountain lions rely on the deer as a food source. The population of each species goes up and down in cycles, but out of phase with each other. A wildlife management research team estimated the respective populations in a particular region every 2 years over a 16-year period, with the results shown in Table 1.

Table 1 Mountain Lion/Deer Populations

Years	0	2	4	6	8	10	12	14	16
Deer	1,272	1,523	1,152	891	1,284	1,543	1,128	917	1,185
Mountain Lions	39	47	63	54	37	48	60	46	40

(A) Deer Population Analysis

1. Enter the data for the deer population for the time interval [0, 16] in a graphing calculator and produce a scatter plot of the data.
2. A function of the form $y = k + A \sin(Bx + C)$ can be used to model these data. Use the data in Table 1 to determine k, A, and B. Use the graph in part 1 to visually estimate C to one decimal place.
3. Plot the data from part 1 and the equation from part 2 in the same viewing window. If necessary, adjust the value of C for a better fit.

4. If your graphing calculator has a sinusoidal regression feature, check your results from parts 2 and 3 by finding and plotting the regression equation.
5. Write a summary of the results, describing fluctuations and cycles of the deer population.

(B) Mountain Lion Population Analysis

1. Enter the data for the mountain lion population for the time interval [0, 16] in a graphing calculator and produce a scatter plot of the data.
2. A function of the form $y = k + A \sin (Bx + C)$ can be used to model these data. Use the data in Table 1 to determine k, A, and B. Use the graph in part 1 to visually estimate C to one decimal place.
3. Plot the data from part 1 and the equation from part 2 in the same viewing window. If necessary, adjust the value of C for a better fit.
4. If your graphing calculator has a sinusoidal regression feature, check your results from parts 2 and 3 by finding and plotting the regression equation.
5. Write a summary of the results, describing fluctuations and cycles of the mountain lion population.

(C) Interrelationship of the Two Populations

1. Discuss the relationship of the maximum predator populations to the maximum prey populations relative to time.
2. Discuss the relationship of the minimum predator populations to the minimum prey populations relative to time.
3. Discuss the dynamics of the fluctuations of the two interdependent populations. What causes the two populations to rise and fall, and why are they out of phase with one another?