CHAPTER 1

Equations and Inequalities

IN this chapter we look at techniques for solving linear equations and inequalities. We introduce complex numbers and examine solution methods for several types of nonlinear equations. Throughout the chapter, we consider applications that can be solved using these techniques. Additional techniques for solving polynomial equations will be discussed in Chapter 4.

SECTIONS

1-1 Linear Equations and Applications

After discussing terminology and solution methods for linear equations, we consider a variety of word problems. The solutions of these word problems involve setting up and solving an appropriate linear equation. The solution of this linear equation then produces a solution to the word problem.

Understanding Basic Terms

An **algebraic equation** is a mathematical statement that relates two algebraic expressions involving at least one variable. Some examples of equations with x as the variable are

$$3x - 2 = 7 \qquad \frac{1}{1 + x} = \frac{x}{x - 2}$$

$$2x^2 - 3x + 5 = 0 \qquad \sqrt{x + 4} = x - 1$$

The **replacement set,** or **domain,** for a variable is defined to be the set of numbers that are permitted to replace the variable.

> **ASSUMPTION On Domains of Variables**
>
> Unless stated to the contrary, we assume that the domain for a variable in an algebraic expression or equation is the set of those real numbers for which the algebraic expressions involving the variable are real numbers.

For example, the domain for the variable x in the expression

$$2x - 4$$

is R, the set of all real numbers, since $2x - 4$ represents a real number for all replacements of x by real numbers. The domain of x in the equation

$$\frac{1}{x} = \frac{2}{x - 3}$$

is the set of all real numbers except 0 and 3. These values are excluded because the expression on the left is not defined for $x = 0$ and the expression on the right is not defined for $x = 3$. Both expressions represent real numbers for all other replacements of x by real numbers.

The **solution set** for an equation is defined to be the set of elements in the domain of the variable that makes the equation true. Each element of the solution set is called a **solution,** or **root,** of the equation. To **solve an equation** is to find the solution set for the equation.

An equation is called an **identity** if the equation is true for all elements from the domain of the variable. An equation is called a **conditional equation** if it is true for certain domain values and false for others. For example,

$$2x - 4 = 2(x - 2) \qquad \text{and} \qquad \frac{5}{x^2 - 3x} = \frac{5}{x(x - 3)}$$

are identities, since both equations are true for all elements from the respective domains of their variables. On the other hand, the equations

$$3x - 2 = 5 \qquad \text{and} \qquad \frac{2}{x - 1} = \frac{1}{x}$$

are conditional equations, since, for example, neither equation is true for the domain value 2.

Knowing what we mean by the solution set of an equation is one thing; finding it is another. We introduce the idea of equivalent equations to help us find solutions. We will call two equations **equivalent** if they both have the same solution set for a given replacement set. To solve an equation, we perform operations on the equation to produce simpler equivalent equations. We stop when we find an equation whose solution is obvious. Then we check this obvious solution in the original equation. Any of the properties of equality given in Theorem 1 can be used to produce equivalent equations.

› THEOREM 1 Properties of Equality

For a, b, and c any real numbers:

1. If $a = b$, then $a + c = b + c$. **Addition Property**
2. If $a = b$, then $a - c = b - c$. **Subtraction Property**
3. If $a = b$ and $c \neq 0$, then $ca = cb$. **Multiplication Property**
4. If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$. **Division Property**
5. If $a = b$, then either may replace the other in any statement without changing the truth or falsity of the statement. **Substitution Property**

Solving Linear Equations

We now turn our attention to methods of solving *first-degree,* or *linear, equations* in one variable.

DEFINITION 1 Linear Equation in One Variable

Any equation that can be written in the form

$$ax + b = 0 \qquad a \neq 0 \qquad \textbf{Standard Form}$$

where a and b are real constants and x is a variable, is called a **linear,** or **first-degree, equation** in one variable.

$5x - 1 = 2(x + 3)$ is a linear equation, since it can be written in the standard form $3x - 7 = 0$.

EXAMPLE 1

Solving a Linear Equation

Solve $5x - 9 = 3x + 7$ and check.

SOLUTION

We use the properties of equality to transform the given equation into an equivalent equation whose solution is obvious.

$$5x - 9 = 3x + 7 \qquad \text{Add 9 to both sides.}$$
$$5x - 9 + 9 = 3x + 7 + 9 \qquad \text{Combine like terms.}$$
$$5x = 3x + 16 \qquad \text{Subtract } 3x \text{ from both sides.}$$
$$5x - 3x = 3x + 16 - 3x \qquad \text{Combine like terms.}$$
$$2x = 16 \qquad \text{Divide both sides by 2.}$$
$$\frac{2x}{2} = \frac{16}{2} \qquad \text{Simplify.}$$
$$x = 8$$

The solution set for this last equation is obvious:

Solution set: $\{8\}$

And since the equation $x = 8$ is equivalent to all the preceding equations in our solution, $\{8\}$ is also the solution set for all these equations, including the original equation. [*Note:* If an equation has only one element in its solution set, we generally use the last equation (in this case, $x = 8$) rather than set notation to represent the solution.]

CHECK

$$5x - 9 = 3x + 7 \qquad \text{Substitute } x = 8.$$
$$5(8) - 9 \stackrel{?}{=} 3(8) + 7 \qquad \text{Simplify each side.}$$
$$40 - 9 \stackrel{?}{=} 24 + 7$$
$$31 \stackrel{\checkmark}{=} 31 \qquad \text{A true statement}$$

MATCHED PROBLEM 1

Solve and check: $7x - 10 = 4x + 5$

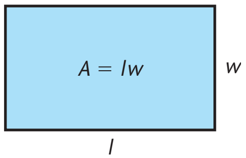


Figure 1 Area of a rectangle.

We frequently encounter equations involving more than one variable. For example, if l and w are the length and width of a rectangle, respectively, the area of the rectangle is given by (see Fig. 1).

$$A = lw$$

Depending on the situation, we may want to solve this equation for l or w. To solve for w, we simply consider A and l to be constants and w to be a variable. Then the equation $A = lw$ becomes a linear equation in w that can be solved easily by dividing both sides by l:

$$w = \frac{A}{l} \qquad l \neq 0$$

EXAMPLE 2 Solving an Equation with More Than One Variable

Solve for P in terms of the other variables: $A = P + Prt$

SOLUTION

$$A = P + Prt \qquad \text{Factor to isolate } P.$$

$$A = P(1 + rt) \qquad \text{Divide both sides by } 1 + rt.$$

$$\frac{A}{1 + rt} = P$$

$$P = \frac{A}{1 + rt} \qquad \text{Restriction: } 1 + rt \neq 0$$

MATCHED PROBLEM 2

Solve for F in terms of C: $C = \frac{5}{9}(F - 32)$

Using a Strategy to Solve Word Problems

A great many practical problems can be solved using algebraic techniques—so many, in fact, that there is no one method of attack that will work for all. However, we can put together a strategy that will help you organize your approach.

STRATEGY FOR SOLVING WORD PROBLEMS

1. Read the problem carefully—several times if necessary—that is, until you understand the problem, know what is to be found, and know what is given.
2. Let one of the unknown quantities be represented by a variable, say x, and try to represent all other unknown quantities in terms of x. This is an important step and must be done carefully.
3. If appropriate, draw figures or diagrams and label known and unknown parts.
4. Look for formulas connecting the known quantities to the unknown quantities.
5. Form an equation relating the unknown quantities to the known quantities.
6. Solve the equation and write answers to *all* questions asked in the problem.
7. Check and interpret all solutions in terms of the original problem—not just the equation found in step 5—because a mistake may have been made in setting up the equation in step 5.

››› EXPLORE-DISCUSS 1

Translate each of the following sentences involving two numbers into an equation.

(A) The first number is 10 more than the second number.

(B) The first number is 15 less than the second number.

(C) The first number is half the second number.

(D) The first number is three times the second number.

(E) Ten times the first number is 15 more than the second number.

The remaining examples in this section contain solutions to a variety of word problems illustrating both the process of setting up word problems and the techniques used to solve the resulting equations. It is suggested that you cover up a solution, try solving the problem yourself, and uncover just enough of a solution to get you going again in case you get stuck. After successfully completing an example, try the matched problem. After completing the section in this way, you will be ready to attempt a fairly large variety of applications.

Solving Number and Geometric Problems

The first examples introduce the process of setting up and solving word problems in a simple mathematical context. Following these, the examples are of a more concrete nature.

EXAMPLE 3 Setting Up and Solving a Word Problem

Find four consecutive even integers such that the sum of the first three exceeds the fourth by 8.

SOLUTION

Let x = the first even integer, then

$$x \qquad x + 2 \qquad x + 4 \qquad \text{and} \qquad x + 6$$

represent four consecutive even integers starting with the even integer x. (Remember, even integers increase by 2.) The phrase "the sum of the first three exceeds the fourth by 8" translates into an equation:

$$\text{Sum of the first three} = \text{Fourth} + \text{Excess}$$

$$x + (x + 2) + (x + 4) = (x + 6) + 8 \qquad \text{Collect like terms.}$$

$$3x + 6 = x + 14 \qquad \text{Subtract 6 from both sides.}$$

$$2x = 8 \qquad \text{Divide both sides by 2.}$$

$$x = 4$$

The four consecutive integers are 4, 6, 8, and 10.

CHECK

$4 + 6 + 8 = 18$	Sum of first three
-8	Excess
10	Fourth

MATCHED PROBLEM 3

Find three consecutive odd integers such that 3 times their sum is 5 more than 8 times the middle one.

››› EXPLORE-DISCUSS 2

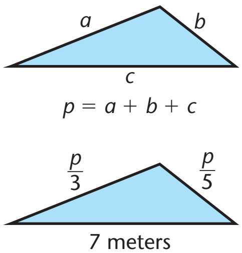
According to property 1 of Theorem 1, multiplying both sides of an equation by a nonzero number always produces an equivalent equation. By what number would you choose to multiply both sides of the following equation to eliminate all the fractions?

$$\frac{x + 1}{3} - \frac{x}{4} = \frac{1}{2}$$

If you did not choose 12, the LCD of all the fractions in this equation, you could still solve the resulting equation, but with more effort. (For a discussion of LCDs and how to find them, see Section R-6.)

EXAMPLE 4 Using a Diagram in the Solution of a Word Problem

If one side of a triangle is one-third the perimeter, the second side is one-fifth the perimeter, and the third side is 7 meters, what is the perimeter of the triangle?



› **Figure 2**

SOLUTION

Let p = the perimeter. Draw a triangle and label the sides, as shown in Figure 2. Then

$$p = a + b + c$$

$$p = \frac{p}{3} + \frac{p}{5} + 7 \qquad \text{Multiply both sides by 15, the LCD. This and the next step usually can be done mentally.}$$

$$15 \cdot p = 15 \cdot \left(\frac{p}{3} + \frac{p}{5} + 7\right) \quad *$$

$$15p = 15 \cdot \frac{p}{3} + 15 \cdot \frac{p}{5} + 15 \cdot 7$$

$$15p = 5p + 3p + 105 \qquad \text{Collect like terms.}$$

$$15p = 8p + 105 \qquad \text{Subtract } 8p \text{ from both sides.}$$

$$7p = 105 \qquad \text{Divide both sides by 7.}$$

$$p = 15$$

The perimeter is 15 meters.

CHECK

$$\frac{p}{3} = \frac{15}{3} = 5 \qquad \text{Side 1}$$

$$\frac{p}{5} = \frac{15}{5} = 3 \qquad \text{Side 2}$$

$$7 \qquad \text{Side 3}$$

$$15 \text{ meters} \qquad \text{Perimeter}$$

MATCHED PROBLEM 4

If one side of a triangle is one-fourth the perimeter, the second side is 7 centimeters, and the third side is two-fifths the perimeter, what is the perimeter?

*Throughout the book, dashed boxes—called **think boxes**—are used to represent steps that may be performed mentally.

>>> **CAUTION** >>>

A very common error occurs about now—students tend to confuse *algebraic expressions* involving fractions with *algebraic equations* involving fractions.

Consider these two problems:

(A) Solve: $\frac{x}{2} + \frac{x}{3} = 10$ **(B)** Add: $\frac{x}{2} + \frac{x}{3} + 10$

The problems look very much alike but are actually very different. To solve the equation in (A) we multiply both sides by 6 (the LCD) to clear the fractions. This works so well for equations that students want to do the same thing for problems like (B). The only catch is that (B) is not an equation, and the multiplication property of equality does not apply. If we multiply (B) by 6, we simply obtain an expression 6 times as large as the original! Compare the following:

(A)

$$\frac{x}{2} + \frac{x}{3} = 10$$

$$6 \cdot \frac{x}{2} + 6 \cdot \frac{x}{3} = 6 \cdot 10$$

$$3x + 2x = 60$$

$$5x = 60$$

$$x = 12$$

(B)

$$\frac{x}{2} + \frac{x}{3} + 10$$

$$= \frac{3 \cdot x}{3 \cdot 2} + \frac{2 \cdot x}{2 \cdot 3} + \frac{6 \cdot 10}{6 \cdot 1}$$

$$= \frac{3x}{6} + \frac{2x}{6} + \frac{60}{6}$$

$$= \frac{5x + 60}{6}$$

› Solving Rate–Time Problems

There are many types of quantity–rate–time problems and distance–rate–time problems. In general, if Q is the quantity of something produced (kilometers, words, parts, and so on) in T units of time (hours, years, minutes, seconds, and so on), then the formulas given in the box are useful.

› QUANTITY–RATE–TIME FORMULAS

$$R = \frac{Q}{T} \qquad \text{Rate} = \frac{\text{Quantity}}{\text{Time}}$$

$$Q = RT \qquad \text{Quantity} = (\text{Rate})(\text{Time})$$

$$T = \frac{Q}{R} \qquad \text{Time} = \frac{\text{Quantity}}{\text{Rate}}$$

If Q is distance D, then

$$R = \frac{D}{T} \qquad D = RT \qquad T = \frac{D}{R}$$

[*Note:* R is an average or uniform rate.]

››› EXPLORE-DISCUSS 3

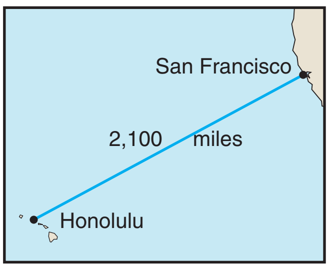
A bus leaves Milwaukee at 12:00 noon and travels due west on Interstate 94 at a constant rate of 55 miles per hour. A passenger that was left behind leaves Milwaukee in a taxicab at 1:00 P.M. in pursuit of the bus. The taxicab travels at a constant rate of 65 miles per hour. Let t represent time in hours after 12:00 noon.

(A) How far has the bus traveled after t hours?

(B) If $t \geq 1$, how far has the taxicab traveled after t hours?

(C) When will the taxicab catch up with the bus?

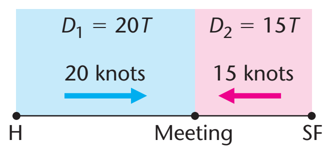
EXAMPLE 5 A Distance–Rate–Time Problem



The distance along a shipping route between San Francisco and Honolulu is 2,100 nautical miles. If one ship leaves San Francisco at the same time another leaves Honolulu, and if the former travels at 15 knots* and the latter at 20 knots, how long will it take the two ships to rendezvous? How far will they be from Honolulu and San Francisco at that time?

SOLUTION

Let T = number of hours until both ships meet. Draw a diagram and label known and unknown parts. Both ships will have traveled the same amount of time when they meet.



$$\begin{pmatrix}\text{Distance ship 1}\\ \text{from Honolulu}\\ \text{travels to}\\ \text{meeting point}\end{pmatrix} + \begin{pmatrix}\text{Distance ship 2}\\ \text{from San Francisco}\\ \text{travels to}\\ \text{meeting point}\end{pmatrix} = \begin{pmatrix}\text{Total distance}\\ \text{from Honolulu}\\ \text{to San Francisco}\end{pmatrix}$$

$$D_1 + D_2 = 2{,}100$$

$$20T + 15T = 2{,}100$$

$$35T = 2{,}100$$

$$T = 60$$

*15 knots means 15 nautical miles per hour. There are 6,076.1 feet in 1 nautical mile, and 5,280 feet in 1 statute mile.

Therefore, it takes 60 hours, or 2.5 days, for the ships to meet.

$$\text{Distance from Honolulu} = 20 \cdot 60 = 1{,}200 \text{ nautical miles}$$
$$\text{Distance from San Francisco} = 15 \cdot 60 = 900 \text{ nautical miles}$$

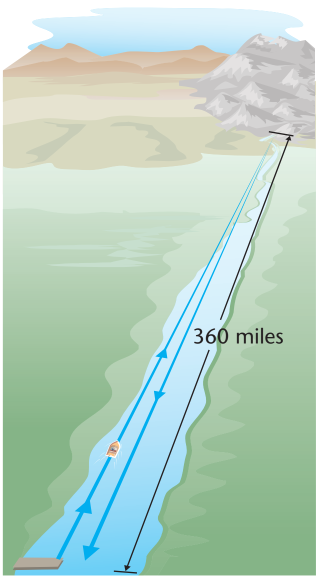
CHECK $1{,}200 + 900 = 2{,}100$ nautical miles

MATCHED PROBLEM 5

An old piece of equipment can print, stuff, and label 38 mailing pieces per minute. A newer model can handle 82 per minute. How long will it take for both pieces of equipment to prepare a mailing of 6,000 pieces? [*Note:* The mathematical form is the same as in Example 5.]

Some equations involving variables in a denominator can be transformed into linear equations. We can proceed in essentially the same way as in Example 5; however, we must exclude any value of the variable that will make a denominator 0. With these values excluded, we can multiply through by the LCD even though it contains a variable, and, according to Theorem 1, the new equation will be equivalent to the old.

EXAMPLE 6 A Distance–Rate–Time Problem



An excursion boat takes 1.5 times as long to go 360 miles up a river as to return. If the boat cruises at 15 miles per hour in still water, what is the rate of the current?

SOLUTION

Let

$$x = \text{Rate of current (in miles per hour)}$$
$$15 - x = \text{Rate of boat upstream}$$
$$15 + x = \text{Rate of boat downstream}$$
$$\text{Time upstream} = (1.5)(\text{Time downstream})$$
$$\frac{\text{Distance upstream}}{\text{Rate upstream}} = (1.5)\frac{\text{Distance downstream}}{\text{Rate downstream}} \qquad \text{Recall: } T = \frac{D}{R}$$
$$\frac{360}{15 - x} = (1.5)\frac{360}{15 + x} \qquad x \neq 15, x \neq -15$$
$$\frac{360}{15 - x} = \frac{540}{15 + x} \qquad \text{Multiply both sides by } (15 - x)(15 + x).$$
$$360(15 + x) = 540(15 - x) \qquad \text{Remove parentheses.}$$
$$5{,}400 + 360x = 8{,}100 - 540x \qquad \text{Add } 540x \text{ to both sides.}$$
$$5{,}400 + 900x = 8{,}100 \qquad \text{Subtract } 5{,}400 \text{ from both sides.}$$
$$900x = 2{,}700 \qquad \text{Divide both sides by } 900.$$
$$x = 3$$

Therefore, the rate of the current is 3 miles per hour. The check is left to the reader.

MATCHED PROBLEM 6

A jetliner takes 1.2 times as long to fly from Paris to New York (3,600 miles) as to return. If the jet cruises at 550 miles per hour in still air, what is the average rate of the wind blowing in the direction of Paris from New York?

EXPLORE-DISCUSS 4

Consider the following solution:

$$\begin{aligned}\frac{x}{x-2} + 2 &= \frac{2x-2}{x-2}\\ x + 2x - 4 &= 2x - 2\\ x &= 2\end{aligned}$$

Is $x = 2$ a root of the original equation? If not, why? Discuss the importance of excluding values that make a denominator 0 when solving equations.

EXAMPLE 7 A Quantity–Rate–Time Problem

An advertising firm has an old computer that can prepare a whole mailing in 6 hours. With the help of a newer model the job is complete in 2 hours. How long would it take the newer model to do the job alone?

SOLUTION

Let x = time (in hours) for the newer model to do the whole job alone.

$$\begin{pmatrix}\text{Part of job completed}\\ \text{in a given length of time}\end{pmatrix} = (\text{Rate})(\text{Time})$$

$$\text{Rate of old model} = \frac{1}{6}\text{ job per hour}$$

$$\text{Rate of new model} = \frac{1}{x}\text{ job per hour}$$

$$\begin{pmatrix}\text{Part of job completed}\\ \text{by old model}\\ \text{in 2 hours}\end{pmatrix} + \begin{pmatrix}\text{Part of job completed}\\ \text{by new model}\\ \text{in 2 hours}\end{pmatrix} = 1 \text{ whole job}$$

$$\begin{pmatrix}\text{Rate of}\\ \text{old model}\end{pmatrix}\begin{pmatrix}\text{Time of}\\ \text{old model}\end{pmatrix} + \begin{pmatrix}\text{Rate of}\\ \text{new model}\end{pmatrix}\begin{pmatrix}\text{Time of}\\ \text{new model}\end{pmatrix} = 1 \qquad \text{Recall: } Q = RT$$

$$\frac{1}{6}(2) + \frac{1}{x}(2) = 1 \qquad x \neq 0$$

$$\frac{1}{3} + \frac{2}{x} = 1 \qquad \text{Multiply both sides by } 3x\text{, the LCD.}$$

$$3x\left(\frac{1}{3}\right) + 3x\left(\frac{2}{x}\right) = 3x$$

$$x + 6 = 3x \qquad \text{Subtract } x \text{ from each side.}$$

$$6 = 2x \qquad \text{Divide both sides by 2.}$$

$$3 = x$$

Therefore, the new computer could do the job alone in 3 hours.

CHECK

Part of job completed by old model in 2 hours $= 2(\frac{1}{6}) = \frac{1}{3}$

Part of job completed by new model in 2 hours $= 2(\frac{1}{3}) = \frac{2}{3}$

Part of job completed by both models in 2 hours $= 1$

MATCHED PROBLEM 7

Two pumps are used to fill a water storage tank at a resort. One pump can fill the tank by itself in 9 hours, and the other can fill it in 6 hours. How long will it take both pumps operating together to fill the tank?

Solving Mixture Problems

A variety of applications can be classified as mixture problems. Even though the problems come from different areas, their mathematical treatment is essentially the same.

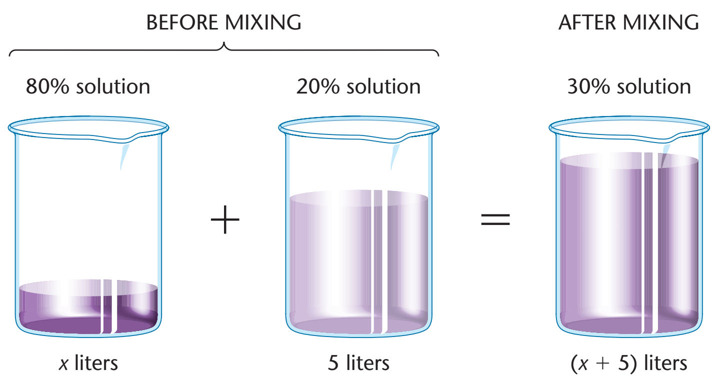
EXAMPLE 8

A Mixture Problem

How many liters of a mixture containing 80% alcohol should be added to 5 liters of a 20% solution to yield a 30% solution?

SOLUTION

Let x = amount of 80% solution used.



$$\begin{pmatrix}\text{Amount of}\\ \text{alcohol in}\\ \text{first solution}\end{pmatrix} + \begin{pmatrix}\text{Amount of}\\ \text{alcohol in}\\ \text{second solution}\end{pmatrix} = \begin{pmatrix}\text{Amount of}\\ \text{alcohol in}\\ \text{mixture}\end{pmatrix}$$

$$\begin{aligned} 0.8x + 0.2(5) &= 0.3(x + 5) \\ 0.8x + 1 &= 0.3x + 1.5 \\ 0.5x &= 0.5 \\ x &= 1 \end{aligned}$$

Add 1 liter of the 80% solution.

CHECK

	Liters of solution	Liters of alcohol	Percent alcohol
First solution	1	$0.8(1) = 0.8$	80
Second solution	5	$0.2(5) = 1$	20
Mixture	6	1.8	$1.8/6 = 0.3$, or 30%

MATCHED PROBLEM 8

A chemical storeroom has a 90% acid solution and a 40% acid solution. How many centiliters of the 90% solution should be added to 50 centiliters of the 40% solution to yield a 50% solution?

ANSWERS TO MATCHED PROBLEMS

1. $x = 5$ **2.** $F = \frac{9}{5}C + 32$ **3.** 3, 5, 7 **4.** 20 centimeters
5. 50 minutes **6.** 50 miles per hour **7.** 3.6 hours **8.** 12.5 centiliters

1-1 Exercises

In Problems 1–16, solve each equation.

1. $4(x + 5) = 6(x - 2)$

2. $3(y - 4) + 2y = 18$

3. $5 + 4(w - 1) = 2w + 2(w + 4)$

4. $4 - 3(t + 2) + t = 5(t - 1) - 7t$

5. $5 - \frac{3a - 4}{5} = \frac{7 - 2a}{2}$

6. $\frac{3b}{7} + \frac{2b - 5}{2} = -4$

7. $\frac{x}{2} + \frac{2x - 1}{3} = \frac{3x + 4}{4}$

8. $\frac{x}{5} + \frac{3x - 1}{2} = \frac{6x + 5}{4}$

9. $0.1(t + 0.5) + 0.2t = 0.3(t - 0.4)$

10. $0.1(w + 0.5) + 0.2w = 0.2(w - 0.4)$

11. $0.35(s + 0.34) + 0.15s = 0.2s - 1.66$

12. $0.35(u + 0.34) - 0.15u = 0.2u - 1.66$

13. $\frac{2}{y} + \frac{5}{2} = 4 - \frac{2}{3y}$

14. $\frac{3 + w}{6w} = \frac{1}{2w} + \frac{4}{3}$

15. $\frac{z}{z - 1} = \frac{1}{z - 1} + 2$

16. $\frac{t}{t - 1} = \frac{2}{t - 1} + 2$

In Problems 17–24, solve each equation.

17. $\frac{2m}{5} + \frac{m-4}{6} = \frac{4m+1}{4} - 2$

18. $\frac{3(n-2)}{5} + \frac{2n+3}{6} = \frac{4n+1}{9} + 2$

19. $1 - \frac{x-3}{x-2} = \frac{2x-3}{x-2}$

20. $\frac{2x-3}{x+1} = 2 - \frac{3x-1}{x+1}$

21. $\frac{6}{y+4} + 1 = \frac{5}{2y+8}$

22. $\frac{4y}{y-3} + 5 = \frac{12}{y-3}$

23. $\frac{3a-1}{a^2+4a+4} - \frac{3}{a^2+2a} = \frac{3}{a}$

24. $\frac{1}{b-5} - \frac{10}{b^2-5b+25} = \frac{1}{b+5}$

*In Problems 25–28, use a calculator to solve each equation to 3 significant digits.**

25. $3.142x - 0.4835(x - 4) = 6.795$

26. $0.0512x + 0.125(x - 2) = 0.725x$

27. $\frac{2.32x}{x-2} - \frac{3.76}{x} = 2.32$

28. $\frac{6.08}{x} + 4.49 = \frac{4.49x}{x+3}$

In Problems 29–36, solve for the indicated variable in terms of the other variables.

29. $a_n = a_1 + (n - 1)d$ for d (arithmetic progressions)

30. $F = \frac{9}{5}C + 32$ for C (temperature scale)

31. $\frac{1}{f} = \frac{1}{d_1} + \frac{1}{d_2}$ for f (simple lens formula)

32. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ for R_1 (electric circuit)

33. $A = 2ab + 2ac + 2bc$ for a (surface area of a rectangular solid)

34. $A = 2ab + 2ac + 2bc$ for c

35. $y = \frac{2x-3}{3x+5}$ for x

36. $x = \frac{3y+2}{y-3}$ for y

In Problems 37 and 38, imagine that the indicated "solutions" were given to you by a student whom you were tutoring in this class. Is the solution right or wrong? If the solution is wrong, explain what is wrong and show a correct solution.

37.
$$\frac{x}{x-3} + 4 = \frac{2x-3}{x-3}$$
$$x + 4x - 12 = 2x - 3$$
$$x = 3$$

38.
$$\frac{x^2+1}{x-1} = \frac{x^2+4x-3}{x-1}$$
$$x^2 + 1 = x^2 + 4x - 3$$
$$x = 1$$

*Appendix A contains a brief discussion of *significant digits*.

In Problems 39–41, solve for x.

39. $\frac{x - \frac{1}{x}}{1 + \frac{1}{x}} = 3$

40. $\frac{x - \frac{1}{x}}{x + 1 - \frac{2}{x}} = 1$

41. $\frac{x + 1 - \frac{2}{x}}{1 - \frac{1}{x}} = x + 2$

42. Solve for y in terms of x: $\frac{y}{1-y} = \left(\frac{x}{1-x}\right)^3$

43. Solve for x in terms of y: $y = \frac{a}{1 + \frac{b}{x+c}}$

44. Let m and n be real numbers with m larger than n. Then there exists a positive real number p such that $m = n + p$. Find the fallacy in the following argument:

$$m = n + p$$
$$(m - n)m = (m - n)(n + p)$$
$$m^2 - mn = mn + mp - n^2 - np$$
$$m^2 - mn - mp = mn - n^2 - np$$
$$m(m - n - p) = n(m - n - p)$$
$$m = n$$

APPLICATIONS

These problems are grouped according to subject area. As before, the most difficult problems are marked with two stars (★★), the moderately difficult problems are marked with one star (★), and the easier problems are not marked.

Numbers

45. Find a number such that 10 less than two-thirds the number is one-fourth the number.

46. Find a number such that 6 more than one-half the number is two-thirds the number.

47. Find four consecutive even integers so that the sum of the first three is 2 more than twice the fourth.

48. Find three consecutive even integers so that the first plus twice the second is twice the third.

Geometry

49. Find the dimensions of a rectangle with a perimeter of 54 meters, if its length is 3 meters less than twice its width.

50. A rectangle 24 meters long has the same area as a square that is 12 meters on a side. What are the dimensions of the rectangle?

51. Find the perimeter of a triangle if one side is 16 feet, another side is two-sevenths the perimeter, and the third side is one-third the perimeter.

52. Find the perimeter of a triangle if one side is 11 centimeters, another side is two-fifths the perimeter, and the third side is one-half the perimeter.

Business and Economics

53. The sale price on a camera after a 20% discount is \$72. What was the price before the discount?

54. A stereo store marks up each item it sells 60% above wholesale price. What is the wholesale price on a cassette player that retails at \$144?

55. One employee of a computer store is paid a base salary of \$2,150 a month plus an 8% commission on all sales over \$7,000 during the month. How much must the employee sell in 1 month to earn a total of \$3,170 for the month?

56. A second employee of the computer store in Problem 55 is paid a base salary of \$1,175 a month plus a 5% commission on all sales during the month.
(A) How much must this employee sell in 1 month to earn a total of \$3,170 for the month?
(B) Determine the sales level where both employees receive the same monthly income. If employees can select either of these payment methods, how would you advise an employee to make this selection?

Earth Science

★**57.** In 1970, Russian scientists began drilling a very deep borehole in the Kola Peninsula. Their goal was to reach a depth of 15 kilometers, but high temperatures in the borehole forced them to stop in 1994 after reaching a depth of 12 kilometers. They found that below 3 kilometers the temperature T increased 2.5°C for each additional 100 meters of depth.
(A) If the temperature at 3 kilometers is 30°C and x is the depth of the hole in kilometers, write an equation using x that will give the temperature T in the hole at any depth beyond 3 kilometers.
(B) What would the temperature be at 12 kilometers?
(C) At what depth (in kilometers) would they reach a temperature of 200°C?

★**58.** Because air is not as dense at high altitudes, planes require a higher ground speed to become airborne. A rule of thumb is 3% more ground speed per 1,000 feet of elevation, assuming no wind and no change in air temperature. (Compute numerical answers to 3 significant digits.)
(A) Let

V_S = Takeoff ground speed at sea level for a particular plane (in miles per hour)

A = Altitude above sea level (in thousands of feet)

V = Takeoff ground speed at altitude A for the same plane (in miles per hour)

Write a formula relating these three quantities.
(B) What takeoff ground speed would be required at Lake Tahoe airport (6,400 feet), if takeoff ground speed at San Francisco airport (sea level) is 120 miles per hour?
(C) If a landing strip at a Colorado Rockies hunting lodge (8,500 feet) requires a takeoff ground speed of 125 miles per hour, what would be the takeoff ground speed in Los Angeles (sea level)?
(D) If the takeoff ground speed at sea level is 135 miles per hour and the takeoff ground speed at a mountain resort is 155 miles per hour, what is the altitude of the mountain resort in thousands of feet?

★★**59.** An earthquake emits a primary wave and a secondary wave. Near the surface of the Earth the primary wave travels at about 5 miles per second, and the secondary wave travels at about 3 miles per second. From the time lag between the two waves arriving at a given seismic station, it is possible to estimate the distance to the quake. Suppose a station measures a time difference of 12 seconds between the arrival of the two waves. How far is the earthquake from the station? (The *epicenter* can be located by obtaining distance bearings at three or more stations.)

★★**60.** A ship using sound-sensing devices above and below water recorded a surface explosion 39 seconds sooner on its underwater device than on its above-water device. If sound travels in air at about 1,100 feet per second and in water at about 5,000 feet per second, how far away was the explosion?

Life Science

61. A naturalist for a fish and game department estimated the total number of trout in a certain lake using the popular capture–mark–recapture technique. She netted, marked, and released 200 trout. A week later, allowing for thorough mixing, she again netted 200 trout and found 8 marked ones among them. Assuming that the ratio of marked trout to the total number in the second sample is the same as the ratio of all marked fish in the first sample to the total trout population in the lake, estimate the total number of fish in the lake.

62. Repeat Problem 61 with a first (marked) sample of 300 and a second sample of 180 with only 6 marked trout.

Chemistry

★**63.** How many gallons of distilled water must be mixed with 50 gallons of 30% alcohol solution to obtain a 25% solution?

★**64.** How many gallons of hydrochloric acid must be added to 12 gallons of a 30% solution to obtain a 40% solution?

★**65.** A chemist mixes distilled water with a 90% solution of sulfuric acid to produce a 50% solution. If 5 liters of distilled water is used, how much 50% solution is produced?

★**66.** A fuel oil distributor has 120,000 gallons of fuel with 0.9% sulfur content, which exceeds pollution control standards of 0.8% sulfur content. How many gallons of fuel oil with a 0.3% sulfur content must be added to the 120,000 gallons to obtain fuel oil that will comply with the pollution control standards?

Rate–Time

★**67.** An old computer can do the weekly payroll in 5 hours. A newer computer can do the same payroll in 3 hours. The old computer starts on the payroll, and after 1 hour the newer computer is brought on-line to work with the older computer until the job is finished. How long will it take both computers working together to finish the job? (Assume the computers operate independently.)

★**68.** One pump can fill a gasoline storage tank in 8 hours. With a second pump working simultaneously, the tank can be filled in 3 hours. How long would it take the second pump to fill the tank operating alone?

★★**69.** The cruising speed of an airplane is 150 miles per hour (relative to the ground). You wish to hire the plane for a 3-hour sightseeing trip. You instruct the pilot to fly north as far as she can and still return to the airport at the end of the allotted time.
(A) How far north should the pilot fly if the wind is blowing from the north at 30 miles per hour?
(B) How far north should the pilot fly if there is no wind?

★**70.** Suppose you are at a river resort and rent a motor boat for 5 hours starting at 7 A.M. You are told that the boat will travel at 8 miles per hour upstream and 12 miles per hour returning. You decide that you would like to go as far up the river as you can and still be back at noon. At what time should you turn back, and how far from the resort will you be at that time?

Music

★**71.** A major chord in music is composed of notes whose frequencies are in the ratio 4:5:6. If the first note of a chord has a frequency of 264 hertz (middle C on the piano), find the frequencies of the other two notes. [*Hint:* Set up two proportions using 4:5 and 4:6.]

★**72.** A minor chord is composed of notes whose frequencies are in the ratio 10:12:15. If the first note of a minor chord is A, with a frequency of 220 hertz, what are the frequencies of the other two notes?

Psychology

73. In an experiment on motivation, Professor Brown trained a group of rats to run down a narrow passage in a cage to receive food in a goal box. He then put a harness on each rat and connected it to an overhead wire attached to a scale. In this way he could place the rat different distances from the food and measure the pull (in grams) of the rat toward the food. He found that the relationship between motivation (pull) and position was given approximately by the equation

$$p = -\tfrac{1}{5}d + 70 \qquad 30 \le d \le 170$$

where pull p is measured in grams and distance d in centimeters. When the pull registered was 40 grams, how far was the rat from the goal box?

74. Professor Brown performed the same kind of experiment as described in Problem 73, except that he replaced the food in the goal box with a mild electric shock. With the same kind of apparatus, he was able to measure the avoidance strength relative to the distance from the object to be avoided. He found that the avoidance strength a (measured in grams) was related to the distance d that the rat was from the shock (measured in centimeters) approximately by the equation

$$a = -\tfrac{4}{3}d + 230 \qquad 30 \le d \le 170$$

If the same rat were trained as described in this problem and in Problem 73, at what distance (to one decimal place) from the goal box would the approach and avoidance strengths be the same? (What do you think the rat would do at this point?)

Puzzle

75. An oil-drilling rig in the Gulf of Mexico stands so that one-fifth of it is in sand, 20 feet of it is in water, and two-thirds of it is in the air. What is the total height of the rig?

76. During a camping trip in the North Woods in Canada, a couple went one-third of the way by boat, 10 miles by foot, and one-sixth of the way by horse. How long was the trip?

★★**77.** After exactly 12 o'clock noon, what time will the hands of a clock be together again?

1-2 Linear Inequalities

- Understanding Inequality Relations and Interval Notation
- Solving Linear Inequalities
- Applying Linear Inequalities to Chemistry

We now turn to the problem of solving linear inequalities in one variable, such as

$$3(x - 5) \geq 5(x + 7) - 10 \qquad \text{and} \qquad -4 \leq 3 - 2x < 7$$

Understanding Inequality Relations and Interval Notation

The preceding mathematical forms involve the **inequality,** or **order, relation**—that is, "less than" and "greater than" relations. Just as we use = to replace the words "is equal to," we use the **inequality symbols** < and > to represent "is less than" and "is greater than," respectively.

While it probably seems obvious to you that

$$2 < 4 \qquad 5 > 0 \qquad 25{,}000 > 1$$

are true, it may not seem as obvious that

$$-4 < -2 \qquad 0 > -5 \qquad -25{,}000 < -1$$

To make the inequality relation precise so that we can interpret it relative to all real numbers, we need a precise definition of the concept.

DEFINITION 1 $a < b$ and $b > a$

For a and b real numbers, we say that ***a* is less than *b*** or ***b* is greater than *a*** and write

$$a < b \qquad \text{or} \qquad b > a$$

if there exists a positive real number p such that $a + p = b$ (or equivalently, $b - a = p$).

We certainly expect that if a positive number is added to *any* real number, the sum is larger than the original. That is essentially what the definition states.

When we write

$$a \leq b$$

we mean $a < b$ or $a = b$ and say ***a* is less than or equal to *b*.** When we write

$$a \geq b$$

we mean $a > b$ or $a = b$ and say ***a* is greater than or equal to *b*.**

The inequality symbols $<$ and $>$ have a very clear geometric interpretation on the real number line. If $a < b$, then a is to the left of b; if $c > d$, then c is to the right of d (Fig. 1).

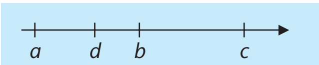


Figure 1
$a < b, c > d$.

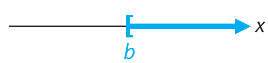
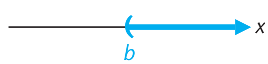
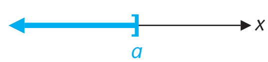
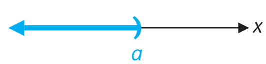
It is an interesting and useful fact that for any two real numbers a and b, either $a < b$, or $a > b$, or $a = b$. This is called the **trichotomy property** of real numbers.

The double inequality $a < x \leq b$ means that $x > a$ and $x \leq b$; that is, x is between a and b, including b but not including a. The set of all real numbers x satisfying the inequality $a < x \leq b$ is called an **interval** and is represented by $(a, b]$. Thus,

$$(a, b] = \{x \mid a < x \leq b\}^*$$

The number a is called the **left endpoint** of the interval, and the symbol "(" indicates that a is not included in the interval. The number b is called the **right endpoint** of the interval, and the symbol "]" indicates that b is included in the interval. An interval is **closed** if it contains its endpoint(s) and **open** if it does not contain any endpoint. Other types of intervals of real numbers are shown in Table 1.

Table 1 Interval Notation

Interval notation	Inequality notation	Line graph	Type
$[a, b]$	$a \leq x \leq b$		Closed
$[a, b)$	$a \leq x < b$		Half-open
$(a, b]$	$a < x \leq b$		Half-open
(a, b)	$a < x < b$		Open
$[b, \infty)$	$x \geq b$		Closed
(b, ∞)	$x > b$		Open
$(-\infty, a]$	$x \leq a$		Closed
$(-\infty, a)$	$x < a$		Open

*In general, $\{x \mid P(x)\}$ represents the set of all x such that statement $P(x)$ is true. To express this set verbally, just read the vertical bar as "such that."

Note that the symbol "∞," read "infinity," used in Table 1 is not a numeral. When we write $[b, \infty)$, we are simply referring to the interval starting at b and continuing indefinitely to the right. We would never write $[b, \infty]$ or $b \le x \le \infty$, because ∞ cannot be used as an endpoint of an interval. The interval $(-\infty, \infty)$ represents the set of real numbers R, since its graph is the entire real number line.

››› CAUTION ›››

It is important to note that

$$5 > x \ge -3 \quad \text{is equivalent to } [-3, 5) \text{ and not to } (5, -3]$$

In interval notation, the smaller number is always written to the left. Thus, it may be useful to rewrite the inequality as $-3 \le x < 5$ before rewriting it in interval notation.

EXAMPLE 1

Graphing Intervals and Inequalities

Write each of the following in inequality notation and graph on a real number line:

(A) $[-2, 3)$ (B) $(-4, 2)$ (C) $[-2, \infty)$ (D) $(-\infty, 3)$

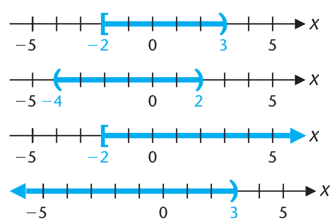
SOLUTIONS

(A) $-2 \le x < 3$

(B) $-4 < x < 2$

(C) $x \ge -2$

(D) $x < 3$



MATCHED PROBLEM 1

Write each of the following in interval notation and graph on a real number line:

(A) $-3 < x \le 3$ (B) $2 \ge x \ge -1$ (C) $x > 1$ (D) $x \le 2$

››› EXPLORE-DISCUSS 1

Example 1C shows the graph of the inequality $x \geq -2$. What is the graph of $x < -2$? What is the corresponding interval? Describe the relationship between these sets.

Since intervals are sets of real numbers, the set operations of *union* and *intersection* are often useful when working with intervals. The **union** of sets A and B, denoted by $A \cup B$, is the set formed by combining all the elements of A and all the elements of B. The **intersection** of sets A and B, denoted by $A \cap B$, is the set of elements of A that are also in B. Symbolically:

› DEFINITION 2 Union and Intersection

Union: $A \cup B = \{x \mid x \text{ is in } A \textbf{ or } x \text{ is in } B\}$

{1, 2, 3} ∪ {2, 3, 4, 5} = {1, 2, 3, 4, 5}

Intersection: $A \cap B = \{x \mid x \text{ is in } A \textbf{ and } x \text{ is in } B\}$

{1, 2, 3} ∩ {2, 3, 4, 5} = {2, 3}

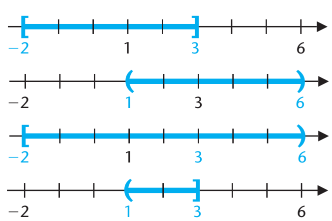
EXAMPLE 2 Graphing Unions and Intersections of Intervals

If $A = [-2, 3]$, $B = (1, 6)$, and $C = (4, \infty)$, graph the indicated sets and write as a single interval, if possible:

(A) $A \cup B$ and $A \cap B$ (B) $A \cup C$ and $A \cap C$

SOLUTIONS

(A)



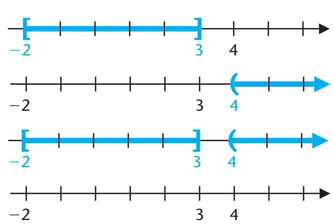
$A = [-2, 3]$

$B = (1, 6)$

$A \cup B = [-2, 6)$

$A \cap B = (1, 3]$

(B)



$A = [-2, 3]$

$C = (4, \infty)$

$A \cup C = [-2, 3] \cup (4, \infty)$

$A \cap C = \varnothing$

MATCHED PROBLEM 2

If $D = [-4, 1)$, $E = (-1, 3]$, and $F = [2, \infty)$, graph the indicated sets and write as a single interval, if possible:

(A) $D \cup E$ (B) $D \cap E$ (C) $E \cup F$ (D) $E \cap F$

EXPLORE-DISCUSS 2

Replace ? with $<$ or $>$ in each of the following.

(A) $-1 \; ? \; 3$ and $2(-1) \; ? \; 2(3)$

(B) $-1 \; ? \; 3$ and $-2(-1) \; ? \; -2(3)$

(C) $12 \; ? \; -8$ and $\frac{12}{4} \; ? \; \frac{-8}{4}$

(D) $12 \; ? \; -8$ and $\frac{12}{-4} \; ? \; \frac{-8}{-4}$

Based on these examples, describe verbally the effect of multiplying both sides of an inequality by a number.

Solving Linear Inequalities

We now turn to the problem of solving linear inequalities in one variable, such as

$$2(2x + 3) < 6(x - 2) + 10 \qquad \text{and} \qquad -3 < 2x + 3 \le 9$$

The **solution set** for an inequality is the set of all values of the variable that make the inequality a true statement. Each element of the solution set is called a **solution** of the inequality. To **solve an inequality** is to find its solution set. Two inequalities are **equivalent** if they have the same solution set for a given replacement set. Just as with equations, we perform operations on inequalities that produce simpler equivalent inequalities, and continue the process until an inequality is reached whose solution is obvious. The properties of inequalities given in Theorem 1 can be used to produce equivalent inequalities.

THEOREM 1 Inequality Properties

For a, b, and c any real numbers:

1. If $a < b$ and $b < c$, then $a < c$. — **Transitive Property**
2. If $a < b$, then $a + c < b + c$. — **Addition Property**
   $-2 < 4$ $\quad 2 + 3 < 4 + 3$
3. If $a < b$, then $a - c < b - c$. — **Subtraction Property**
   $-2 < 4$ $\quad -2 - 3 < 4 - 3$
4. If $a < b$ and c is positive, then $ca < cb$.
   $-2 < 4$ $\quad 3(-2) < 3(4)$
5. If $a < b$ and c is negative, then $ca > cb$.
   $-2 < 4$ $\quad (-3)(-2) > (-3)(4)$

   Multiplication Property (Note difference between 4 and 5.)
6. If $a < b$ and c is positive, then $\frac{a}{c} < \frac{b}{c}$.
   $-2 < 4$ $\quad \frac{-2}{2} < \frac{4}{2}$
7. If $a < b$ and c is negative, then $\frac{a}{c} > \frac{b}{c}$.
   $-2 < 4$ $\quad \frac{-2}{-2} > \frac{4}{-2}$

   Division Property (Note difference between 6 and 7.)

Similar properties hold if each inequality sign is reversed, or if $<$ is replaced with $\leq$ and $>$ is replaced with $\geq$. Thus, we find that we can perform essentially the same operations on inequalities that we perform on equations. When working with inequalities, however, we have to be particularly careful of the use of the multiplication and division properties.

The order of the inequality reverses if we multiply or divide both sides of an inequality statement by a negative number.

EXPLORE-DISCUSS 3

Properties of equality are easily summarized: We can add, subtract, multiply, or divide both sides of an equation by any nonzero real number to produce an equivalent equation. Write a similar summary for the properties of inequalities.

Now let's see how the inequality properties are used to solve linear inequalities. Examples 3, 4, and 5 will illustrate the process.

EXAMPLE 3 Solving a Linear Inequality

Solve and graph: $2(2x + 3) - 10 < 6(x - 2)$

SOLUTION

$$2(2x + 3) - 10 < 6(x - 2)$$ Remove parentheses.

$$4x + 6 - 10 < 6x - 12$$ Combine like terms.

$$4x - 4 < 6x - 12$$ Add 4 to both sides.

$$4x - 4 + 4 < 6x - 12 + 4$$

$$4x < 6x - 8$$ Subtract $6x$ from both sides.

$$4x - 6x < 6x - 8 - 6x$$

$$-2x < -8$$ Divide both sides by -2. Note that order reverses since -2 is negative

$$\frac{-2x}{-2} > \frac{-8}{-2}$$

$$x > 4 \quad \text{or} \quad (4, \infty)$$ Solution set

Graph of solution set

MATCHED PROBLEM 3

Solve and graph: $3(x - 1) \geq 5(x + 2) - 5$

EXAMPLE 4 Solving a Linear Inequality Involving Fractions

Solve and graph: $\frac{2x - 3}{4} + 6 \geq 2 + \frac{4x}{3}$

SOLUTION

$$\frac{2x - 3}{4} + 6 \geq 2 + \frac{4x}{3}$$ Multiply both sides by 12, the LCD.

$$12 \cdot \frac{2x - 3}{4} + 12 \cdot 6 \geq 12 \cdot 2 + 12 \cdot \frac{4x}{3}$$

$$3(2x - 3) + 72 \geq 24 + 4(4x)$$ Remove parentheses.

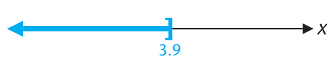
$$6x - 9 + 72 \geq 24 + 16x$$ Collect like terms.

$$6x + 63 \geq 24 + 16x$$ Subtract 63 from both sides.

$$6x \geq -39 + 16x$$ Subtract $16x$ from both sides.

$$-10x \geq -39$$ Order reverses when both sides are divided by -10, a negative number.

$$x \leq 3.9 \quad \text{or} \quad (-\infty, 3.9]$$



MATCHED PROBLEM 4

Solve and graph: $\frac{4x - 3}{3} + 8 < 6 + \frac{3x}{2}$

EXAMPLE 5

Solving a Double Inequality

Solve and graph: $-3 \le 4 - 7x < 18$

SOLUTION

We proceed as before, except we try to isolate x in the middle with a coefficient of 1.

$$-3 \le 4 - 7x < 18$$ Subtract 4 from each member.

$$-3 - 4 \le 4 - 7x - 4 < 18 - 4$$

$$-7 \le -7x < 14$$ Divide each member by -7 and reverse each inequality.

$$\frac{-7}{-7} \ge \frac{-7x}{-7} > \frac{14}{-7}$$

$$1 \ge x > -2 \quad \text{or} \quad -2 < x \le 1 \quad \text{or} \quad (-2, 1]$$

MATCHED PROBLEM 5

Solve and graph: $-3 < 7 - 2x \le 7$

› Applying Linear Inequalities to Chemistry

EXAMPLE 6

Chemistry

In a chemistry experiment, a solution of hydrochloric acid is to be kept between 30°C and 35°C—that is, $30 \le C \le 35$. What is the range in temperature in degrees Fahrenheit if the Celsius/Fahrenheit conversion formula is $C = \frac{5}{9}(F - 32)$?

SOLUTION

$$30 \le C \le 35$$ Replace C with $\frac{5}{9}(F - 32)$.

$$30 \le \frac{5}{9}(F - 32) \le 35$$ Multiply each member by $\frac{9}{5}$.

$$\frac{9}{5} \cdot 30 \le \frac{9}{5} \cdot \frac{5}{9}(F - 32) \le \frac{9}{5} \cdot 35$$

$$54 \le F - 32 \le 63$$ Add 32 to each member.

$$54 + 32 \le F - 32 + 32 \le 63 + 32$$

$$86 \le F \le 95$$

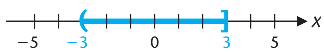
The range of the temperature is from 86°F to 95°F, inclusive.

MATCHED PROBLEM 6

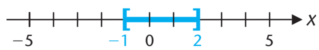
A film developer is to be kept between 68°F and 77°F—that is, $68 \le F \le 77$. What is the range in temperature in degrees Celsius if the Celsius/Fahrenheit conversion formula is $F = \frac{9}{5}C + 32$?

ANSWERS TO MATCHED PROBLEMS

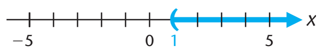
1. (A) $(-3, 3]$



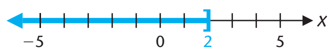
(B) $[-1, 2]$



(C) $(1, \infty)$



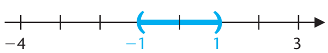
(D) $(-\infty, 2]$



2. (A)

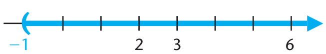
$D \cup E = [-4, 3]$

(B)



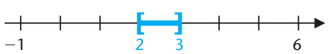
$D \cap E = (-1, 1)$

(C)



$E \cup F = (-1, \infty)$

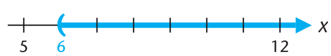
(D)



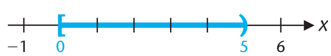
$E \cap F = [2, 3]$

3. $x \le -4$ or $(-\infty, -4]$

4. $x > 6$ or $(6, \infty)$



5. $5 > x \ge 0$ or $0 \le x < 5$ or $[0, 5)$



6. $20 \le C \le 25$: the range in temperature is from 20°C to 25°C

1-2 Exercises

In Problems 1–6, rewrite in inequality notation and graph on a real number line.

1. $[-8, 7]$ **2.** $(-4, 8)$ **3.** $[-6, 6)$

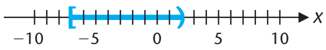
4. $(-3, 3]$ **5.** $[-6, \infty)$ **6.** $(-\infty, 7)$

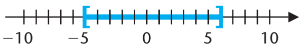
In Problems 7–12, rewrite in interval notation and graph on a real number line.

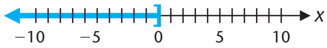
7. $-2 < x \le 6$ **8.** $-5 \le x \le 5$ **9.** $-7 < x < 8$

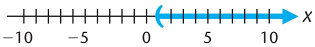
10. $-4 \le x < 5$ **11.** $x \le -2$ **12.** $x > 3$

In Problems 13–16, write in interval and inequality notation.

13. 

14. 

15. 

16. 

In Problems 17–24, replace each ? with > or < to make the resulting statement true.

17. $12 \, ? \, 6$ and $12 + 5 \, ? \, 6 + 5$

18. $-4 \, ? \, -2$ and $-4 - 7 \, ? \, -2 - 7$

19. $-6 \, ? \, -8$ and $-6 - 3 \, ? \, -8 - 3$

20. $4 \, ? \, 9$ and $4 + 2 \, ? \, 9 + 2$

21. $2 \, ? \, -1$ and $-2(2) \, ? \, -2(-1)$

22. $-3 \, ? \, 2$ and $4(-3) \, ? \, 4(2)$

23. $2 \, ? \, 6$ and $\frac{2}{2} \, ? \, \frac{6}{2}$

24. $-10 \, ? \, -15$ and $\frac{-10}{5} \, ? \, \frac{-15}{5}$

In Problems 25–38, solve and graph.

25. $7x - 8 < 4x + 7$

26. $4x + 8 \geq x - 1$

27. $3 - x \geq 5(3 - x)$

28. $2(x - 3) + 5 < 5 - x$

29. $\frac{N}{-2} > 4$

30. $\frac{M}{-3} \leq -2$

31. $-5t < -10$

32. $-7n \geq 21$

33. $3 - m < 4(m - 3)$

34. $2(1 - u) \geq 5u$

35. $-2 - \frac{B}{4} \leq \frac{1 + B}{3}$

36. $\frac{y - 3}{4} - 1 > \frac{y}{2}$

37. $-4 < 5t + 6 \leq 21$

38. $2 \leq 3m - 7 < 14$

In Problems 39–50, graph the indicated set and write as a single interval, if possible.

39. $(-5, 5) \cup [4, 7]$

40. $(-5, 5) \cap [4, 7]$

41. $[-1, 4) \cap (2, 6]$

42. $[-1, 4) \cup (2, 6]$

43. $(-\infty, 1) \cup (-2, \infty)$

44. $(-\infty, 1) \cap (2, \infty)$

45. $(-\infty, -1) \cup [3, 7)$

46. $(1, 6] \cup [9, \infty)$

47. $[2, 3] \cup (1, 5)$

48. $[2, 3] \cap (1, 5)$

49. $(-\infty, 4) \cup (-1, 6]$

50. $(-3, 2) \cup [0, \infty)$

In Problems 51–66, solve and graph.

51. $\frac{q}{7} - 3 > \frac{q - 4}{3} + 1$

52. $\frac{p}{3} - \frac{p - 2}{2} \leq \frac{p}{4} - 4$

53. $\frac{2x}{5} - \frac{1}{2}(x - 3) \leq \frac{2x}{3} - \frac{3}{10}(x + 2)$

54. $\frac{2}{3}(x + 7) - \frac{x}{4} > \frac{1}{2}(3 - x) + \frac{x}{6}$

55. $-4 \leq \frac{9}{5}x + 32 \leq 68$

56. $-1 \leq \frac{2}{3}A + 5 \leq 11$

57. $-12 < \frac{3}{4}(2 - x) \leq 24$

58. $24 \leq \frac{2}{3}(x - 5) < 36$

59. $16 < 7 - 3x \leq 31$

60. $-1 \leq 9 - 2x < 5$

61. $-6 < -\frac{2}{5}(1 - x) \leq 4$

62. $15 \leq 7 - \frac{2}{5}x \leq 21$

63. $0.1(x - 7) < 0.8 - 0.05x$

64. $0.4(x + 5) > 0.3x + 17$

65. $0.3x - 2.04 \geq 0.04(x + 1)$

66. $0.02x - 5.32 \leq 0.5(x - 2)$

Problems 67–72 are calculus-related. For what real number(s) x does each expression represent a real number?

67. $\sqrt{1 - x}$

68. $\sqrt{x + 5}$

69. $\sqrt{3x + 5}$

70. $\sqrt{7 - 2x}$

71. $\frac{1}{\sqrt[4]{2x + 3}}$

72. $\frac{1}{\sqrt[4]{5 - 6x}}$

73. What can be said about the signs of the numbers a and b in each case?
(A) $ab > 0$ (B) $ab < 0$
(C) $\frac{a}{b} > 0$ (D) $\frac{a}{b} < 0$

74. What can be said about the signs of the numbers a, b, and c in each case?
(A) $abc > 0$ (B) $\frac{ab}{c} < 0$
(C) $\frac{a}{bc} > 0$ (D) $\frac{a^2}{bc} < 0$

75. Replace each question mark with < or >, as appropriate:
(A) If $a - b = 1$, then $a \, ? \, b$.
(B) If $u - v = -2$, then $u \, ? \, v$.

76. For what p and q is $p + q < p - q$?

77. If both a and b are negative numbers and b/a is greater than 1, then is $a - b$ positive or negative?

78. If both a and b are positive numbers and b/a is greater than 1, then is $a - b$ positive or negative?

79. Indicate true (T) or false (F):
(A) If $p > q$ and $m > 0$, then $mp < mq$.
(B) If $p < q$ and $m < 0$, then $mp > mq$.
(C) If $p > 0$ and $q < 0$, then $p + q > q$.

80. Assume that $m > n > 0$; then

$$\begin{aligned} mn &> n^2 \\ mn - m^2 &> n^2 - m^2 \\ m(n - m) &> (n + m)(n - m) \\ m &> n + m \\ 0 &> n \end{aligned}$$

But it was assumed that $n > 0$. Find the error.

Prove each inequality property in Problems 81–84, given a, b, and c are arbitrary real numbers.

81. If $a < b$, then $a + c < b + c$.

82. If $a < b$, then $a - c < b - c$.

83. (A) If $a < b$ and c is positive, then $ca < cb$.
(B) If $a < b$ and c is negative, then $ca > cb$.

84. (A) If $a < b$ and c is positive, then $\frac{a}{c} < \frac{b}{c}$.
(B) If $a < b$ and c is negative, then $\frac{a}{c} > \frac{b}{c}$.

APPLICATIONS

Write all your answers using inequality notation.

85. **EARTH SCIENCE** In 1970, Russian scientists began drilling a very deep borehole in the Kola Peninsula. Their goal was to reach a depth of 15 kilometers, but high temperatures in the borehole forced them to stop in 1994 after reaching a depth of 12 kilometers. They found that the approximate temperature x kilometers below the surface of the Earth is given by

$$T = 30 + 25(x - 3) \qquad 3 \le x \le 12$$

where T is temperature in degrees Celsius. At what depth is the temperature between 150°C and 250°C, inclusive?

86. **EARTH SCIENCE** As dry air moves upward it expands, and in so doing it cools at a rate of about 5.5°F for each 1,000-foot rise up to about 40,000 feet. If the ground temperature is 70°F, then the temperature T at height h is given approximately by $T = 70 - 0.0055h$. For what range in altitude will the temperature be between 26°F and −40°F, inclusive?

87. **BUSINESS AND ECONOMICS** An electronics firm is planning to market a new graphing calculator. The fixed costs are $650,000 and the variable costs are $47 per calculator. The wholesale price of the calculator will be $63. For the company to make a profit, it is clear that revenues must be greater than costs.
(A) How many calculators must be sold for the company to make a profit?
(B) How many calculators must be sold for the company to break even?
(C) Discuss the relationship between the results in parts A and B.

88. **BUSINESS AND ECONOMICS** A video game manufacturer is planning to market a new version of its game machine. The fixed costs are $550,000 and the variable costs are $120 per machine. The wholesale price of the machine will be $140.
(A) How many game machines must be sold for the company to make a profit?
(B) How many game machines must be sold for the company to break even?
(C) Discuss the relationship between the results in parts A and B.

89. **BUSINESS AND ECONOMICS** The electronics firm in Problem 87 finds that rising prices for parts increases the variable costs to $50.50 per calculator.
(A) Discuss possible strategies the company might use to deal with this increase in costs.
(B) If the company continues to sell the calculators for $63, how many must they sell now to make a profit?
(C) If the company wants to start making a profit at the same production level as before the cost increase, how much should they increase the wholesale price?

90. **BUSINESS AND ECONOMICS** The video game manufacturer in Problem 88 finds that unexpected programming problems increases the fixed costs to $660,000.
(A) Discuss possible strategies the company might use to deal with this increase in costs.
(B) If the company continues to sell the game machines for $140, how many must they sell now to make a profit?
(C) If the company wants to start making a profit at the same production level as before the cost increase, how much should they increase the wholesale price?

91. **ENERGY** If the power demands in a 110-volt electric circuit in a home vary between 220 and 2,750 watts, what is the range of current flowing through the circuit? ($W = EI$, where W = Power in watts, E = Pressure in volts, and I = Current in amperes.)

92. **PSYCHOLOGY** A person's IQ is given by the formula

$$\text{IQ} = \frac{\text{MA}}{\text{CA}}100$$

where MA is mental age and CA is chronological age. If

$$80 \le \text{IQ} \le 140$$

for a group of 12-year-old children, find the range of their mental ages.

1-3 Absolute Value in Equations and Inequalities

- Relating Absolute Value and Distance
- Solving Absolute Value Equations and Inequalities
- Using Absolute Value to Solve Radical Inequalities

We can express the distance between two points on a number line using the concept of *absolute value.* As a result, absolute values often appear in equations and inequalities that are associated with distance. In this section we define absolute value and we show how to solve equations and inequalities that involve absolute value.

Relating Absolute Value and Distance

We start with a geometric definition of absolute value. If a is the coordinate of a point on a real number line, then the distance from the origin to a is represented by $|a|$ and is referred to as the **absolute value** of a. Thus, $|5| = 5$, since the point with coordinate 5 is five units from the origin, and $|-6| = 6$, since the point with coordinate -6 is six units from the origin (Fig. 1).

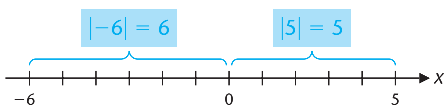


Figure 1
Absolute value.

Symbolically, and more formally, we define absolute value as follows:

DEFINITION 1 Absolute Value

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

$|-3| = -(-3) = 3$
$|4| = 4$

[*Note:* $-x$ is positive if x is negative.]

Both the geometric and nongeometric definitions of absolute value are useful, as will be seen in the material that follows. Remember:

The absolute value of a number is never negative.

EXAMPLE 1 Absolute Value of a Real Number

(A) $|\pi - 3| = \pi - 3$ Since $\pi \approx 3.14$, $\pi - 3$ is positive.

(B) $|3 - \pi| = -(3 - \pi) = \pi - 3$ Since $3 - \pi$ is negative

MATCHED PROBLEM 1

Write without the absolute value sign:

(A) $|8|$ (B) $|\sqrt[3]{9} - 2|$ (C) $|-\sqrt{2}|$ (D) $|2 - \sqrt[3]{9}|$

Following the same reasoning used in Example 1, the Theorem 1 can be proved (see Problem 77 in Exercises 1-3).

THEOREM 1 For all real numbers *a* and *b*,

$$|b - a| = |a - b|$$

We use this result in defining the distance between two points on a real number line.

DEFINITION 2 Distance Between Points *A* and *B*

Let A and B be two points on a real number line with coordinates a and b, respectively. The **distance between** $\boldsymbol{A}$ **and** $\boldsymbol{B}$ is given by

$$d(A, B) = |b - a|$$

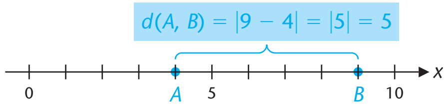
This distance is also called the **length of the line segment** joining A and B.

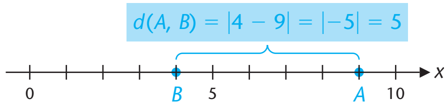
EXAMPLE 2 Distance Between Points on a Number Line

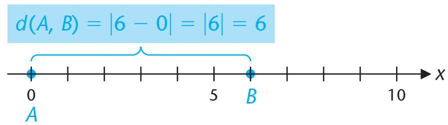
Find the distance between points A and B with coordinates a and b, respectively, as given.

(A) $a = 4, b = 9$ (B) $a = 9, b = 4$ (C) $a = 0, b = 6$

SOLUTIONS

(A)


(B)


(C)


It should be clear, since $|b - a| = |a - b|$, that

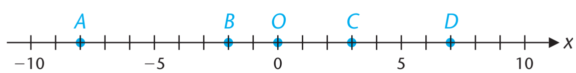
$$d(A, B) = d(B, A)$$

Hence, in computing the distance between two points on a real number line, it does not matter how the two points are labeled—point A can be to the left or to the right of point B. Note also that if A is at the origin O, then

$$d(O, B) = |b - 0| = |b|$$

MATCHED PROBLEM 2

Use the number line shown here to find the indicated distances.



(A) $d(C, D)$ (B) $d(D, C)$ (C) $d(A, B)$

(D) $d(A, C)$ (E) $d(O, A)$ (F) $d(D, A)$

› Solving Absolute Value Equations and Inequalities

The interplay between algebra and geometry is an important tool when working with equations and inequalities involving absolute value. For example, the algebraic statement

$$|x - 1| = 2$$

can be interpreted geometrically as stating that the distance from x to 1 is 2.

»» EXPLORE-DISCUSS 1

Write geometric interpretations of the following algebraic statements:

(A) $|x - 1| < 2$ **(B)** $0 < |x - 1| < 2$ **(C)** $|x - 1| > 2$

EXAMPLE 3 Solving Absolute Value Problems Geometrically

Interpret geometrically, solve, and graph. Write solutions in both inequality and interval notation, where appropriate.

(A) $|x - 3| = 5$ (B) $|x - 3| < 5$

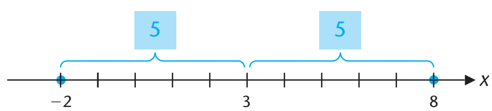
(C) $0 < |x - 3| < 5$ (D) $|x - 3| > 5$

SOLUTIONS

(A) Geometrically, $|x - 3|$ represents the distance between x and 3. Thus, in $|x - 3| = 5$, x is a number whose distance from 3 is 5. That is,

$$x = 3 \pm 5 = -2 \quad \text{or} \quad 8$$

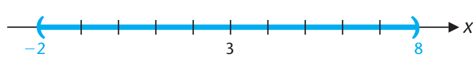
The solution set is $\{-2, 8\}$. **This *is not* interval notation.**



(B) Geometrically, in $|x - 3| < 5$, x is a number whose distance from 3 is less than 5; that is,

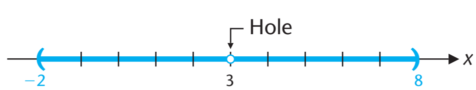
$$-2 < x < 8$$

The solution set is $(-2, 8)$. **This *is* interval notation.**



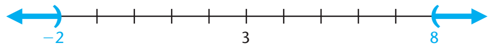
(C) The form $0 < |x - 3| < 5$ is frequently encountered in calculus and more advanced mathematics. Geometrically, x is a number whose distance from 3 is less than 5, but x cannot equal 3. Thus,

$$-2 < x < 8 \quad x \neq 3 \quad \text{or} \quad (-2, 3) \cup (3, 8)$$



(D) Geometrically, in $|x - 3| > 5$, x is a number whose distance from 3 is greater than 5; that is,

$$x < -2 \qquad \text{or} \qquad x > 8 \qquad (-\infty, -2) \cup (8, \infty)$$



››› CAUTION ›››

Do not confuse solutions like

$$-2 < x \qquad \text{and} \qquad x < 8$$

which can also be written as

$$-2 < x < 8 \qquad \text{or} \qquad (-2, 8)$$

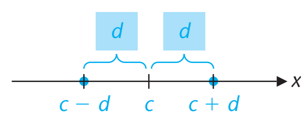
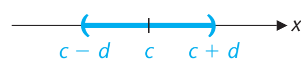
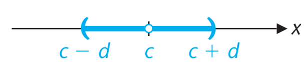
with solutions like

$$x < -2 \qquad \text{or} \qquad x > 8$$

which cannot be written as a double inequality or as a single interval.

We summarize the preceding results in Table 1.

Table 1 Geometric Interpretation of Absolute Value Equations and Inequalities

Form ($d > 0$)	Geometric interpretation	Solution	Graph
$\lvert x - c\rvert = d$	Distance between x and c is equal to d.	$\{c - d, c + d\}$	
$\lvert x - c\rvert < d$	Distance between x and c is less than d.	$(c - d, c + d)$	
$0 < \lvert x - c\rvert < d$	Distance between x and c is less than d, but $x \neq c$.	$(c - d, c) \cup (c, c + d)$	
$\lvert x - c\rvert > d$	Distance between x and c is greater than d.	$(-\infty, c - d) \cup (c + d, \infty)$	

MATCHED PROBLEM 3

Interpret geometrically, solve, and graph. Write solutions in both inequality and interval notation, where appropriate.

(A) $|x + 2| = 6$ (B) $|x + 2| < 6$

(C) $0 < |x + 2| < 6$ (D) $|x + 2| > 6$

[*Hint:* $|x + 2| = |x - (-2)|$.]

EXAMPLE 4

Interpreting Verbal Statements Algebraically

Express each verbal statement as an absolute value equation or inequality.

(A) x is 4 units from 2.

(B) y is less than 3 units form -5.

(C) t is no more than 5 units from 7.

(D) w is no less than 2 units form -1.

SOLUTIONS

(A) $d(x, 2) = |x - 2| = 4$

(B) $d(y, -5) = |y + 5| < 3$

(C) $d(t, 7) = |t - 7| \leq 5$

(D) $d(w, -1) = |w + 1| \geq 2$

MATCHED PROBLEM 4

Express each verbal statement as an absolute value equation or inequality.

(A) x is 6 units from 5.

(B) y is less than 7 units from -6.

(C) w is no less than 3 units from -2.

(D) t is no more than 4 units from 3.

EXPLORE-DISCUSS 2

Describe the set of numbers that satisfies each of the following:

(A) $2 > x > 1$ **(B)** $2 > x < 1$

(C) $2 < x > 1$ **(D)** $2 < x < 1$

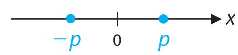
Explain why it is never necessary to use double inequalities with inequality symbols pointing in different directions. Standard mathematical notation requires that all inequality symbols in an expression must point in the same direction.

Reasoning geometrically as before (noting that $|x| = |x - 0|$) leads to Theorem 2.

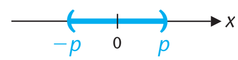
THEOREM 2 Properties of Equations and Inequalities Involving $|x|$

For $p > 0$:

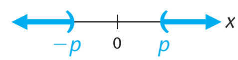
1. $|x| = p$ is equivalent to $x = p$ or $x = -p$.



2. $|x| < p$ is equivalent to $-p < x < p$.



3. $|x| > p$ is equivalent to $x < -p$ or $x > p$.



If we replace x in Theorem 2 with $ax + b$, we obtain the more general Theorem 3.

THEOREM 3 Properties of Equations and Inequalities Involving $|ax + b|$

For $p > 0$:

1. $|ax + b| = p$ is equivalent to $ax + b = p$ or $ax + b = -p$.

2. $|ax + b| < p$ is equivalent to $-p < ax + b < p$.

3. $|ax + b| > p$ is equivalent to $ax + b < -p$ or $ax + b > p$.

EXAMPLE 5 Solving Absolute Value Problems

Solve, and write solutions in both inequality and interval notation, where appropriate.

(A) $|3x + 5| = 4$ (B) $|x| < 5$ (C) $|2x - 1| < 3$ (D) $|7 - 3x| \leq 2$

SOLUTIONS

(A)
$$\begin{aligned} |3x + 5| &= 4 \\ 3x + 5 &= \pm 4 \\ 3x &= -5 \pm 4 \\ x &= \frac{-5 \pm 4}{3} \\ x &= -3, -\tfrac{1}{3} \\ \text{or} \quad &\{-3, -\tfrac{1}{3}\} \end{aligned}$$

(B)
$$\begin{aligned} &|x| < 5 \\ &-5 < x < 5 \\ &\text{or } (-5, 5) \end{aligned}$$

(C)
$$\begin{aligned} &|2x - 1| < 3 \\ &-3 < 2x - 1 < 3 \\ &-2 < 2x < 4 \\ &-1 < x < 2 \\ &\text{or } (-1, 2) \end{aligned}$$

(D)
$$\begin{aligned} &|7 - 3x| \leq 2 \\ &-2 \leq 7 - 3x \leq 2 \\ &-9 \leq -3x \leq -5 \\ &3 \geq x \geq \tfrac{5}{3} \\ &\tfrac{5}{3} \leq x \leq 3 \\ &\text{or } [\tfrac{5}{3}, 3] \end{aligned}$$

MATCHED PROBLEM 5

Solve, and write solutions in both inequality and interval notation, where appropriate.

(A) $|2x - 1| = 8$ (B) $|x| \leq 7$ (C) $|3x + 3| \leq 9$ (D) $|5 - 2x| < 9$

EXAMPLE 6 Solving Absolute Value Inequalities

Solve, and write solutions in both inequality and interval notation.

(A) $|x| > 3$ (B) $|2x - 1| \geq 3$ (C) $|7 - 3x| > 2$

SOLUTIONS

(A) $|x| > 3$ — Use Theorem 2 to remove absolute value.

$x < -3$ or $x > 3$ — Inequality notation

$(-\infty, -3) \cup (3, \infty)$ — Interval notation

(B) $|2x - 1| \geq 3$ — Use Theorem 3 to remove absolute value.

$2x - 1 \leq -3$ or $2x - 1 \geq 3$ — Add 1 to both sides.

$2x \leq -2$ or $2x \geq 4$ — Divide both sides by 2.

$x \leq -1$ or $x \geq 2$ — Inequality notation

$(-\infty, -1] \cup [2, \infty)$ — Interval notation

(C) $|7 - 3x| > 2$ — Use Theorem 3 to remove absolute value.

$7 - 3x < -2$ or $7 - 3x > 2$ — Subtract 7 from both sides.

$-3x < -9$ or $-3x > -5$ — Divide both sides by -3 and reverse the order of the inequality.

$x > 3$ or $x < \frac{5}{3}$ — Inequality notation

$(-\infty, \frac{5}{3}) \cup (3, \infty)$ — Interval notation

MATCHED PROBLEM 6

Solve, and write solutions in both inequality and interval notation.

(A) $|x| \geq 5$ (B) $|4x - 3| > 5$ (C) $|6 - 5x| > 16$

EXAMPLE 7 An Absolute Value Problem with Two Cases

Solve: $|x + 4| = 3x - 8$

SOLUTION

Theorem 3 does not apply directly, since we do not know that $3x - 8$ is positive. However, we can use the definition of absolute value and two cases: $x + 4 \geq 0$ and $x + 4 < 0$.

Case 1. $x + 4 \geq 0$ (that is, $x \geq -4$)
For this case, the possible values of x are in the set $\{x \mid x \geq -4\}$.

$$|x + 4| = 3x - 8$$
$$x + 4 = 3x - 8 \qquad |a| = a \text{ for } a \geq 0$$
$$-2x = -12$$
$$x = 6 \qquad \text{A solution, since 6 is among the possible values of } x$$

The check is left to the reader.

Case 2. $x + 4 < 0$ (that is, $x < -4$)
In this case, the possible values of x are in the set $\{x \mid x < -4\}$.

$$|x + 4| = 3x - 8$$
$$-(x + 4) = 3x - 8 \qquad |a| = -a \text{ for } a < 0$$
$$-x - 4 = 3x - 8$$
$$-4x = -4$$
$$x = 1 \qquad \text{Not a solution, since 1 is not among the possible values of } x$$

Combining both cases, we see that the only solution is $x = 6$.

CHECK As a final check, we substitute $x = 6$ and $x = 1$ in the original equation.

$$\begin{aligned} |x + 4| &= 3x - 8 \\ |6 + 4| &\overset{?}{=} 3(6) - 8 \\ 10 &\overset{\checkmark}{=} 10 \end{aligned} \qquad \begin{aligned} |x + 4| &= 3x - 8 \\ |1 + 4| &\overset{?}{=} 3(1) - 8 \\ 5 &\neq -5 \end{aligned}$$

MATCHED PROBLEM 7

Solve: $|3x - 4| = x + 5$

Using Absolute Value to Solve Radical Inequalities

In Section R-3, we found that if x is positive or 0, then

$$\sqrt{x^2} = x$$

If x is negative, however, we must write

$$\sqrt{x^2} = -x \qquad \sqrt{(-2)^2} = -(-2) = 2$$

Thus, for x any real number,

$$\sqrt{x^2} = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

But this is exactly how we defined $|x|$ at the beginning of this section (see Definition 1). Thus, for x any real number,

$$\sqrt{x^2} = |x| \tag{1}$$

EXAMPLE 8 Solving a Radical Inequality

Use equation (1) to solve:

$$\sqrt{(x - 2)^2} \leq 5$$

Write your answers in both inequality and interval notation.

SOLUTION

$$\begin{aligned} \sqrt{(x - 2)^2} &\leq 5 && \text{Use equation (1).} \\ |x - 2| &\leq 5 && \text{Use Theorem 3.} \\ -5 \leq x - 2 &\leq 5 && \text{Add 2 to each member.} \\ -3 \leq x &\leq 7 && \text{Inequality notation} \\ \text{or } &[-3, 7] && \text{Interval notation} \end{aligned}$$

MATCHED PROBLEM 8

Use equation (1) to solve:

$$\sqrt{(x+2)^2} < 3$$

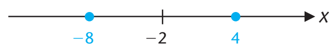
Write your answers in both inequality and interval notation.

ANSWERS TO MATCHED PROBLEMS

1. (A) 8 (B) $\sqrt[3]{9} - 2$ (C) $\sqrt{2}$ (D) $\sqrt[3]{9} - 2$

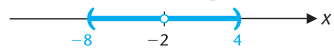
2. (A) 4 (B) 4 (C) 6 (D) 11 (E) 8 (F) 15

3. (A) x is a number whose distance from -2 is 6.
$x = -8, 4$ or $\{-8, 4\}$

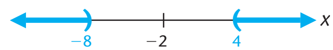


(B) x is a number whose distance from -2 is less than 6.
$-8 < x < 4$ or $(-8, 4)$

(C) x is a number whose distance from -2 is less than 6, but x cannot equal -2.
$-8 < x < 4$, $x \neq -2$, or $(-8, -2) \cup (-2, 4)$



(D) x is a number whose distance from -2 is greater than 6.
$x < -8$ or $x > 4$, or $(-\infty, -8) \cup (4, \infty)$



4. (A) $|x - 5| = 6$ (B) $|y + 6| < 7$ (C) $|w + 2| \geq 3$ (D) $|t - 3| \leq 4$

5. (A) $x = -\frac{7}{2}, \frac{9}{2}$ or $\{-\frac{7}{2}, \frac{9}{2}\}$ (B) $-7 \leq x \leq 7$ or $[-7, 7]$ (C) $-4 \leq x \leq 2$ or $[-4, 2]$ (D) $-2 < x < 7$ or $(-2, 7)$

6. (A) $x \leq -5$ or $x \geq 5$, or $(-\infty, -5] \cup [5, \infty)$ (B) $x < -\frac{1}{2}$ or $x > 2$, or $(-\infty, -\frac{1}{2}) \cup (2, \infty)$ (C) $x < -2$ or $x > \frac{22}{5}$, or $(-\infty, -2) \cup (\frac{22}{5}, \infty)$

7. $x = -\frac{1}{4}, \frac{9}{2}$ or $\{-\frac{1}{4}, \frac{9}{2}\}$ **8.** $-5 < x < 1$ or $(-5, 1)$

1-3 Exercises

In Problems 1–8, simplify, and write without absolute value signs. Do not replace radicals with decimal approximations.

1. $|\sqrt{5}|$ **2.** $|-\frac{3}{4}|$

3. $|(-6) - (-2)|$ **4.** $|(-2) - (-6)|$

5. $|5 - \sqrt{5}|$ **6.** $|\sqrt{7} - 2|$

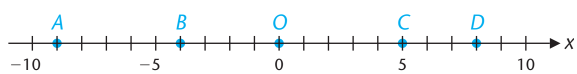
7. $|\sqrt{5} - 5|$ **8.** $|2 - \sqrt{7}|$

In Problems 9–12, find the distance between points A and B with coordinates a and b respectively, as given.

9. $a = -7, b = 5$ **10.** $a = 3, b = 12$

11. $a = 5, b = -7$ **12.** $a = -9, b = -17$

In Problems 13–18, use the number line shown to find the indicated distances.



13. $d(B, O)$ **14.** $d(A, B)$ **15.** $d(O, B)$

16. $d(B, A)$ **17.** $d(B, C)$ **18.** $d(D, C)$

Write each of the statements in Problems 19–28 as an absolute value equation or inequality.

19. x is 4 units from 3.

20. y is 3 units from 1.

21. m is 5 units from -2.

22. n is 7 units from -5.

23. x is less than 5 units from 3.

24. z is less than 8 units from -2.

25. p is more than 6 units from -2.

26. c is no greater than 7 units from -3.

27. q is no less than 2 units from 1.

28. d is no more than 4 units from 5.

In Problems 29–40, solve, interpret geometrically, and graph. When applicable, write answers using both inequality notation and interval notation.

29. $|y - 5| = 3$ **30.** $|t - 3| = 4$ **31.** $|y - 5| < 3$

32. $|t - 3| < 4$ **33.** $|y - 5| > 3$ **34.** $|t - 3| > 4$

35. $|u + 8| = 3$ **36.** $|x + 1| = 5$ **37.** $|u + 8| \leq 3$

38. $|x + 1| \leq 5$ **39.** $|u + 8| \geq 3$ **40.** $|x + 1| \geq 5$

In Problems 41–58, solve each equation or inequality. When applicable, write answers using both inequality notation and interval notation.

41. $|3x - 7| \leq 4$ **42.** $|5y + 2| \geq 8$

43. $|4 - 2t| > 6$ **44.** $|10 + 4s| < 6$

45. $|7m + 11| = 3$ **46.** $|4 - 5n| \leq 8$

47. $|\frac{1}{2}w - \frac{3}{4}| < 2$ **48.** $|\frac{1}{3}z + \frac{5}{6}| = 1$

49. $|0.2u + 1.7| \geq 0.5$ **50.** $|0.5v - 2.5| > 1.6$

51. $|\frac{9}{5}C + 32| < 31$ **52.** $|\frac{5}{9}(F - 32)| < 40$

53. $\sqrt{x^2} < 2$ **54.** $\sqrt{m^2} > 3$

55. $\sqrt{(1 - 3t)^2} \leq 2$ **56.** $\sqrt{(3 - 2x)^2} < 5$

57. $\sqrt{(2t - 3)^2} > 3$ **58.** $\sqrt{(3m + 5)^2} \geq 4$

In Problems 59–62, solve and write answers in inequality notation. Round decimals to three significant digits.

59. $|2.25 - 1.02x| \leq 1.64$

60. $|0.962 - 0.292x| \leq 2.52$

61. $|21.7 - 11.3x| = 15.2$

62. $|195 - 55.5x| = 315$

Problems 63–66 are calculus-related. Solve and graph. Write each solution using interval notation.

63. $0 < |x - 3| < 0.1$ **64.** $0 < |x - 5| < 0.01$

65. $0 < |x - c| < d$ **66.** $0 < |x - 4| < d$

In Problems 67–74, for what values of x does each hold?

67. $|x - 2| = 2x - 7$ **68.** $|x + 4| = 3x + 8$

69. $|3x + 5| = 2x + 6$ **70.** $|7 - 2x| = 5 - x$

71. $|x| + |x + 3| = 3$ **72.** $|x| - |x - 5| = 5$

73. $|2x + 7| - |6 - 3x| = 8$

74. $|3x + 1| + |3 - 2x| = 11$

75. What are the possible values of $\frac{x}{|x|}$?

76. What are the possible values of $\frac{|x - 1|}{x - 1}$?

77. Prove that $|b - a| = |a - b|$ for all real numbers a and b.

78. Prove that $|x|^2 = x^2$ for all real numbers x.

79. Prove that the average of two numbers is between the two numbers; that is, if $m < n$, then

$$m < \frac{m + n}{2} < n$$

80. Prove that for $m < n$,

$$d\left(m, \frac{m + n}{2}\right) = d\left(\frac{m + n}{2}, n\right)$$

81. Prove that $|-m| = |m|$.

82. Prove that $|m| = |n|$ if and only if $m = n$ or $m = -n$.

83. Prove that for $n \neq 0$,

$$\left|\frac{m}{n}\right| = \frac{|m|}{|n|}$$

84. Prove that $|mn| = |m||n|$.

85. Prove that $-|m| \leq m \leq |m|$.

86. Prove the **triangle inequality:**

$$|m + n| \leq |m| + |n|$$

Hint: Use Problem 85 to show that

$$-|m| - |n| \leq m + n \leq |m| + |n|$$

87. If a and b are real numbers, prove that the maximum of a and b is given by

$$\max(a, b) = \tfrac{1}{2}[a + b + |a - b|]$$

88. Prove that the minimum of a and b is given by

$$\min(a, b) = \tfrac{1}{2}[a + b - |a - b|]$$

APPLICATIONS

89. STATISTICS Inequalities of the form

$$\left|\frac{x - m}{s}\right| < n$$

occur frequently in statistics. If $m = 45.4$, $s = 3.2$, and $n = 1$, solve for x.

90. STATISTICS Repeat Problem 89 for $m = 28.6$, $s = 6.5$, and $n = 2$.

★**91.** BUSINESS The daily production P in an automobile assembly plant is within 20 units of 500 units. Express the daily production as an absolute value inequality.

★**92.** CHEMISTRY In a chemical process, the temperature T is to be kept within 10°C of 200°C. Express this restriction as an absolute value inequality.

93. APPROXIMATION The area A of a region is approximately equal to 12.436. The error in this approximation is less than 0.001. Describe the possible values of this area both with an absolute value inequality and with interval notation.

94. APPROXIMATION The volume V of a solid is approximately equal to 6.94. The error in this approximation is less than 0.02. Describe the possible values of this volume both with an absolute value inequality and with interval notation.

★**95.** SIGNIFICANT DIGITS If $N = 2.37$ represents a measurement, then we assume an accuracy of 2.37 ± 0.005. Express the accuracy assumption using an absolute value inequality.

★**96.** SIGNIFICANT DIGITS If $N = 3.65 \times 10^{-3}$ is a number from a measurement, then we assume an accuracy of $3.65 \times 10^{-3} \pm 5 \times 10^{-6}$. Express the accuracy assumption using an absolute value inequality.

1-4 Complex Numbers

- Understanding Complex Number Terminology
- Performing Operations with Complex Numbers
- Relating Complex Numbers and Radicals
- Solving Equations Involving Complex Numbers

The Pythagoreans (500–275 B.C.) found that the simple equation

$$x^2 = 2 \tag{1}$$

had no rational number solutions. If equation (1) were to have a solution, then a new kind of number had to be invented—an irrational number. The irrational numbers $\sqrt{2}$ and $-\sqrt{2}$ are both solutions to equation (1). Irrational numbers were not put on a firm mathematical foundation until the nineteenth century. The rational and irrational numbers together constitute the real number system.

Is there any need to consider another number system? Yes, if we want the simple equation

$$x^2 = -1$$

to have a solution. If x is any real number, then $x^2 \geq 0$. Thus, $x^2 = -1$ cannot have any real number solutions. Once again a new type of number must be invented, a number whose square can be negative. These new numbers are among the numbers called *complex numbers*. The complex numbers evolved over a long period of time, but, like the real numbers, it was not until the nineteenth century that they were placed on a firm mathematical foundation (see Table 1).

Table 1 Brief History of Complex Numbers

Approximate date	Person	Event
50	Heron of Alexandria	First recorded encounter of a square root of a negative number
850	Mahavira of India	Said that a negative has no square root, since it is not a square
1545	Cardano of Italy	Solutions to cubic equations involved square roots of negative numbers.
1637	Descartes of France	Introduced the terms *real* and *imaginary*
1748	Euler of Switzerland	Used i for $\sqrt{-1}$
1832	Gauss of Germany	Introduced the term *complex number*

Understanding Complex Number Terminology

We start the development of the complex number system by defining a complex number and listing some special terms related to complex numbers.

DEFINITION 1 Complex Number

A **complex number** is a number of the form

$$a + bi \qquad \textbf{Standard Form}$$

where a and b are real numbers and i is called the **imaginary unit.**

The imaginary unit i introduced in Definition 1 is not a real number. It is a special symbol used in the representation of the elements in this new complex number system.

Some examples of complex numbers are

$$3 - 2i \qquad \tfrac{1}{2} + 5i \qquad 2 - \tfrac{1}{3}i$$
$$0 + 3i \qquad 5 + 0i \qquad 0 + 0i$$

The notation $3 - 2i$ is shorthand for $3 + (-2)i$.

Particular kinds of complex numbers are given special names as follows:

DEFINITION 2 Special Terms

i		**Imaginary Unit**
$a + bi$	a and b real numbers	**Complex Number**
$a + bi$	$b \neq 0$	**Imaginary Number**
$0 + bi = bi$	$b \neq 0$	**Pure Imaginary Number**
bi		**Imaginary Part of $a + bi$**
$a + 0i = a$		**Real Number**
a		**Real Part of $a + bi$**
$0 = 0 + 0i$		**Zero**
$a - bi$		**Conjugate of $a + bi$**

EXAMPLE 1 Complex Numbers

Identify the real part, the imaginary part, and the conjugate of each of the following numbers:

(A) $3 - 2i$ (B) $2 + 5i$ (C) $7i$ (D) 6

SOLUTIONS

(A) Real part: 3; imaginary part: $-2i$; conjugate: $3 + 2i$

(B) Real part: 2; imaginary part: $5i$; conjugate: $2 - 5i$

(C) Real part: 0; imaginary part: $7i$; conjugate: $-7i$

(D) Real part: 6; imaginary part: 0; conjugate: 6

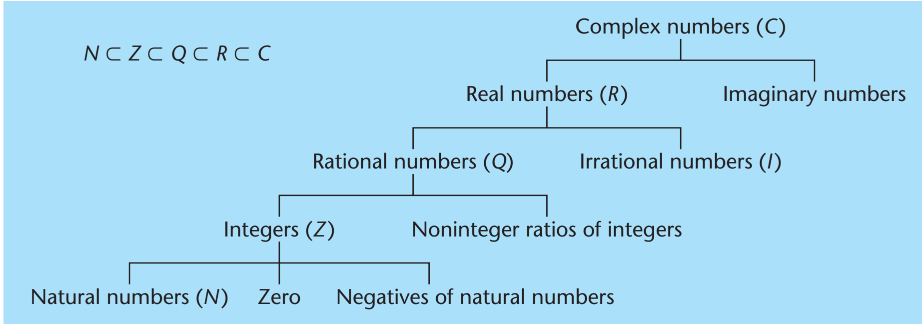
MATCHED PROBLEM 1

Identify the real part, the imaginary part, and the conjugate of each of the following numbers:

(A) $6 + 7i$ (B) $-3 - 8i$ (C) $-4i$ (D) -9

In Definition 2, notice that we identify a complex number of the form $a + 0i$ with the real number a, a complex number of the form $0 + bi$, $b \neq 0$, with the **pure imaginary number** bi, and the complex number $0 + 0i$ with the real number 0. Thus, a real number is also a complex number, just as a rational number is also a real number. Any complex number that is not a real number is called an **imaginary number.** If we combine the set of all real numbers with the set of all imaginary numbers, we obtain ***C*, the set of complex numbers.** The relationship of the complex number system to the other number systems we have studied is shown in Figure 1.

Figure 1
Complex numbers and important subsets



Performing Operations with Complex Numbers

To use complex numbers, we must know how to add, subtract, multiply, and divide them. We start by defining equality, addition, and multiplication.

DEFINITION 3 Equality and Basic Operations

1. **Equality:** $a + bi = c + di$ if and only if $a = c$ and $b = d$
2. **Addition:** $(a + bi) + (c + di) = (a + c) + (b + d)i$
3. **Multiplication:** $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

In Section R-1 we listed the basic properties of the real number system. Using Definition 3, it can be shown that the complex number system possesses the same properties. That is,

1. Addition and multiplication of complex numbers are commutative and associative operations.
2. There is an additive identity and a multiplicative identity for complex numbers.
3. Every complex number has an additive inverse or negative.
4. Every nonzero complex number has a multiplicative inverse or reciprocal.
5. Multiplication distributes over addition.

As a consequence of these properties, we can manipulate complex number symbols of the form $a + bi$ just like we manipulate binomials of the form $a + bx$, as long as we remember that i is a special symbol for the imaginary unit, not for a real number. Thus, you will not have to memorize the definitions of addition and multiplication of complex numbers. We now discuss these operations and some of their properties. Others will be considered in Exercises 1-4.

EXAMPLE 2 Addition of Complex Numbers

Carry out each operation and express the answer in standard form:

(A) $(2 - 3i) + (6 + 2i)$ (B) $(-5 + 4i) + (0 + 0i)$

SOLUTIONS

(A) We could apply the definition of addition directly, but it is easier to use complex number properties.

$$
\begin{aligned}
(2 - 3i) + (6 + 2i) &= 2 - 3i + 6 + 2i && \text{Remove parentheses.}\\
&= (2 + 6) + (-3 + 2)i && \text{Combine like terms.}\\
&= 8 - i
\end{aligned}
$$

(B)
$$
\begin{aligned}
(-5 + 4i) + (0 + 0i) &= -5 + 4i + 0 + 0i\\
&= -5 + 4i
\end{aligned}
$$

MATCHED PROBLEM 2

Carry out each operation and express the answer in standard form:

(A) $(3 + 2i) + (6 - 4i)$ (B) $(0 + 0i) + (7 - 5i)$

Example 2B and Matched Problem 2B illustrate the following general result: For any complex number $a + bi$,

$$(a + bi) + (0 + 0i) = (0 + 0i) + (a + bi) = a + bi$$

Thus, $0 + 0i$ is the **additive identity** or **zero** for the complex numbers. We anticipated this result in Definition 2 when we identified the complex number $0 + 0i$ with the real number 0.

The **additive inverse** or **negative** of $a + bi$ is $-a - bi$ because

$$(a + bi) + (-a - bi) = (-a - bi) + (a + bi) = 0$$

EXAMPLE 3 Negation and Subtraction

Carry out each operation and express the answer in standard form:

(A) $-(4 - 5i)$ (B) $(7 - 3i) - (6 + 2i)$ (C) $(-2 + 7i) + (2 - 7i)$

SOLUTIONS

(A) $-(4 - 5i) = (-1)(4 - 5i) = -4 + 5i$

(B) $(7 - 3i) - (6 + 2i) = 7 - 3i - 6 - 2i$ Remove parentheses.

$= 1 - 5i$ Combine like terms.

(C) $(-2 + 7i) + (2 - 7i) = -2 + 7i + 2 - 7i = 0$

MATCHED PROBLEM 3

Carry out each operation and express the answer in standard form:

(A) $-(-3 + 2i)$ (B) $(3 - 5i) - (1 - 3i)$ (C) $(-4 + 9i) + (4 - 9i)$

Now we turn our attention to multiplication. First, we use the definition of multiplication to see what happens to the complex unit i when it is squared:

$$\begin{aligned} i^2 &= (\overset{a}{0} + \overset{b}{1}i)(\overset{c}{0} + \overset{d}{1}i) \\ &= (\overset{a}{0} \cdot \overset{c}{0} - \overset{b}{1} \cdot \overset{d}{1}) + (\overset{a}{0} \cdot \overset{d}{1} + \overset{b}{1} \cdot \overset{c}{0})i \\ &= -1 + 0i \\ &= -1 \end{aligned}$$

Thus, we have proved that

$$i^2 = -1$$

Just as was the case with addition and subtraction, multiplication of complex numbers can be carried out by using the properties of complex numbers rather than the definition of multiplication. We just replace i^2 with -1 each time it occurs.

EXAMPLE 4 Multiplying Complex Numbers

Carry out each operation and express the answer in standard form:

(A) $(2 - 3i)(6 + 2i)$ (B) $1(3 - 5i)$

(C) $i(1 + i)$ (D) $(3 + 4i)(3 - 4i)$

SOLUTIONS

(A) $(2 - 3i)(6 + 2i) = 2(6 + 2i) - 3i(6 + 2i)$

$= 12 + 4i - 18i - 6i^2$ Replace i^2 with -1.

$= 12 - 14i - 6(-1)$

$= 18 - 14i$

(B) $1(3 - 5i) = 1 \cdot 3 - 1 \cdot 5i = 3 - 5i$

(C) $i(1 + i) = i + i^2 = i - 1 = -1 + i$

(D) $(3 + 4i)(3 - 4i) = 9 - 12i + 12i - 16i^2$
$= 9 + 16 = 25$

MATCHED PROBLEM 4

Carry out each operation and express the answer in standard form:

(A) $(5 + 2i)(4 - 3i)$ (B) $3(-2 + 6i)$

(C) $i(2 - 3i)$ (D) $(2 + 3i)(2 - 3i)$

For any complex number $a + bi$,

$$1(a + bi) = (a + bi)1 = a + bi$$

(see Example 4B). Thus, 1 is the **multiplicative identity** for complex numbers, just as it is for real numbers.

Earlier we stated that every nonzero complex number has a multiplicative inverse or reciprocal. We will denote this as a fraction, just as we do with real numbers. Thus,

$$\frac{1}{a + bi} \quad \text{is the } \textbf{reciprocal} \text{ of} \quad a + bi \qquad a + bi \neq 0$$

The following important property of the conjugate of a complex number is used to express reciprocals and quotients in standard form.

THEOREM 1 Product of a Complex Number and Its Conjugate

$$(a + bi)(a - bi) = a^2 + b^2 \qquad \text{A real number}$$

EXAMPLE 5

Reciprocals and Quotients

Write each expression in standard form:

(A) $\dfrac{1}{2 + 3i}$ (B) $\dfrac{7 - 3i}{1 + i}$

SOLUTIONS

(A) Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{1}{2+3i} = \frac{1}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{2-3i}{4-9i^2} = \frac{2-3i}{4+9}$$

$$= \frac{2-3i}{13} = \frac{2}{13} - \frac{3}{13}i$$

This answer can be checked by multiplication:

CHECK

$$(2+3i)\left(\frac{2}{13} - \frac{3}{13}i\right) = \frac{4}{13} - \frac{6}{13}i + \frac{6}{13}i - \frac{9}{13}i^2$$

$$= \frac{4}{13} + \frac{9}{13} = 1$$

(B) $$\frac{7-3i}{1+i} = \frac{7-3i}{1+i} \cdot \frac{1-i}{1-i} = \frac{7-7i-3i+3i^2}{1-i^2}$$

$$= \frac{4-10i}{2} = 2-5i$$

CHECK

$$(1+i)(2-5i) = 2-5i+2i-5i^2 = 7-3i$$

MATCHED PROBLEM 5

Carry out each operation and express the answer in standard form:

(A) $\frac{1}{4+2i}$ (B) $\frac{6+7i}{2-i}$

EXAMPLE 6 Combined Operations

Carry out the indicated operations and write each answer in standard form:

(A) $(3-2i)^2 - 6(3-2i) + 13$ (B) $\frac{2-3i}{2i}$

SOLUTIONS

(A) $$(3-2i)^2 - 6(3-2i) + 13 = 9 - 12i + 4i^2 - 18 + 12i + 13$$

$$= 9 - 12i - 4 - 18 + 12i + 13$$

$$= 0$$

(B) If a complex number is divided by a pure imaginary number, we can make the denominator real by multiplying numerator and denominator by i.

$$\frac{2-3i}{2i}\cdot\frac{i}{i}=\frac{2i-3i^2}{2i^2}=\frac{2i+3}{-2}=-\frac{3}{2}-i$$

MATCHED PROBLEM 6

Carry out the indicated operations and write each answer in standard form:

(A) $(3+2i)^2-6(3+2i)+13$ (B) $\dfrac{4-i}{3i}$

EXPLORE-DISCUSS 1

Natural number powers of i take on particularly simple forms:

$$\begin{aligned} &i && i^5=i^4\cdot i=(1)i=i \\ &i^2=-1 && i^6=i^4\cdot i^2=1(-1)=-1 \\ &i^3=i^2\cdot i=(-1)i=-i && i^7=i^4\cdot i^3=1(-i)=-i \\ &i^4=i^2\cdot i^2=(-1)(-1)=1 && i^8=i^4\cdot i^4=1\cdot 1=1 \end{aligned}$$

In general, what are the possible values for i^n, n a natural number? Explain how you could easily evaluate i^n for any natural number n. Then evaluate each of the following:

(A) i^{17} **(B)** i^{24} **(C)** i^{38} **(D)** i^{47}

Relating Complex Numbers and Radicals

Recall that we say that a is a square root of b if $a^2=b$. If x is a positive real number, then x has two square roots, the principal square root, denoted by $\sqrt{x}$, and its negative, $-\sqrt{x}$ (Section R-3). If x is a negative real number, then x still has two square roots, but now these square roots are imaginary numbers.

DEFINITION 4 Principal Square Root of a Negative Real Number

The **principal square root of a negative real number,** denoted by $\sqrt{-a}$, where a is positive, is defined by

$$\sqrt{-a}=i\sqrt{a} \qquad \sqrt{-3}=i\sqrt{3} \qquad \sqrt{-9}=i\sqrt{9}=3i$$

The other square root of $-a$, $a>0$, is $-\sqrt{-a}=-i\sqrt{a}$.

Note in Definition 4 that we wrote $i\sqrt{a}$ and $i\sqrt{3}$ in place of the standard forms $\sqrt{a}i$ and $\sqrt{3}i$. We follow this convention whenever it appears that i might accidentally slip under a radical sign ($\sqrt{a}i \neq \sqrt{ai}$, but $\sqrt{a}i = i\sqrt{a}$). Definition 4 is motivated by the fact that

$$(i\sqrt{a})^2 = i^2a = -a$$

EXAMPLE 7 Complex Numbers and Radicals

Write in standard form:

(A) $\sqrt{-4}$ (B) $4 + \sqrt{-5}$ (C) $\dfrac{-3 - \sqrt{-5}}{2}$ (D) $\dfrac{1}{1 - \sqrt{-9}}$

SOLUTIONS

(A) $\sqrt{-4} = i\sqrt{4} = 2i$ (B) $4 + \sqrt{-5} = 4 + i\sqrt{5}$

(C) $\dfrac{-3 - \sqrt{-5}}{2} = \dfrac{-3 - i\sqrt{5}}{2} = -\dfrac{3}{2} - \dfrac{\sqrt{5}}{2}i$

(D) $\dfrac{1}{1 - \sqrt{-9}} = \dfrac{1}{1 - 3i} = \dfrac{1 \cdot (1 + 3i)}{(1 - 3i) \cdot (1 + 3i)}$

$$= \frac{1 + 3i}{1 - 9i^2} = \frac{1 + 3i}{10} = \frac{1}{10} + \frac{3}{10}i$$

MATCHED PROBLEM 7

Write in standard form:

(A) $\sqrt{-16}$ (B) $5 + \sqrt{-7}$ (C) $\dfrac{-5 - \sqrt{-2}}{2}$ (D) $\dfrac{1}{3 - \sqrt{-4}}$

››› EXPLORE-DISCUSS 2

From Theorem 1 in Section R-3, we know that if a and b are positive real numbers, then

$$\sqrt{a}\sqrt{b} = \sqrt{ab} \qquad (2)$$

Thus, we can evaluate expressions like $\sqrt{9}\sqrt{4}$ two ways:

$$\sqrt{9}\sqrt{4} = \sqrt{(9)(4)} = \sqrt{36} = 6 \quad \text{and} \quad \sqrt{9}\sqrt{4} = (3)(2) = 6$$

Evaluate each of the following two ways. Is equation (2) a valid property to use in all cases?

(A) $\sqrt{9}\sqrt{-4}$ **(B)** $\sqrt{-9}\sqrt{4}$ **(C)** $\sqrt{-9}\sqrt{-4}$

››› CAUTION ›››

Note that in Example 7D, we wrote $1 - \sqrt{-9} = 1 - 3i$ before proceeding with the simplification. Writing a complex number in standard form is a necessary step because some of the properties of radicals that are true for real numbers turn out not to be true for complex numbers. In particular, for positive real numbers a and b,

$$\sqrt{a}\sqrt{b} = \sqrt{ab} \qquad \text{but} \qquad \sqrt{-a}\sqrt{-b} \neq \sqrt{(-a)(-b)}$$

(See Explore-Discuss 2.)

› Solving Equations Involving Complex Numbers

EXAMPLE 8 Equations Involving Complex Numbers

(A) Solve for real numbers x and y:

$$(3x + 2) + (2y - 4)i = -4 + 6i$$

(B) Solve for complex number z:

$$(3 + 2i)z - 3 + 6i = 8 - 4i$$

SOLUTIONS

(A) Equate the real and imaginary parts of each side of the equation to form two equations:

Real Parts	**Imaginary Parts**
$3x + 2 = -4$	$2y - 4 = 6$
$3x = -6$	$2y = 10$
$x = -2$	$y = 5$

(B)
$$(3 + 2i)z - 3 + 6i = 8 - 4i \qquad \text{Add } 3 - 6i \text{ to both sides.}$$
$$(3 + 2i)z = 11 - 10i \qquad \text{Divide both sides by } 3 + 2i.$$
$$z = \frac{11 - 10i}{3 + 2i} \qquad \text{Multiply numerator and denominator by } 3 - 2i.$$
$$= \frac{(11 - 10i)(3 - 2i)}{(3 + 2i)(3 - 2i)} \qquad \text{Simplify.}$$
$$= \frac{13 - 52i}{13}$$
$$= 1 - 4i$$

A check is left to the reader.

MATCHED PROBLEM 8

(A) Solve for real numbers x and y:

$$(2y - 7) + (3x + 4)i = 1 + i$$

(B) Solve for complex number z:

$$(1 + 3i)z + 4 - 5i = 3 + 2i$$

Early resistance to these new numbers is suggested by the words used to name them: *complex* and *imaginary*. In spite of this early resistance, complex numbers have come into widespread use in both pure and applied mathematics. They are used extensively, for example, in electrical engineering, physics, chemistry, statistics, and aeronautical engineering. Our first use of them will be in connection with solutions of second-degree equations in Section 1-5.

ANSWERS TO MATCHED PROBLEMS

1. (A) Real part: 6; imaginary part: $7i$; conjugate: $6 - 7i$ (B) Real part: -3; imaginary part: $-8i$; conjugate: $-3 + 8i$ (C) Real part: 0; imaginary part: $-4i$; conjugate: $4i$ (D) Real part: -9; imaginary part: 0; conjugate: -9
2. (A) $9 - 2i$ (B) $7 - 5i$
3. (A) $3 - 2i$ (B) $2 - 2i$ (C) 0
4. (A) $26 - 7i$ (B) $-6 + 18i$ (C) $3 + 2i$ (D) 13
5. (A) $\frac{1}{5} - \frac{1}{10}i$ (B) $1 + 4i$
6. (A) 0 (B) $-\frac{1}{3} - \frac{4}{3}i$
7. (A) $4i$ (B) $5 + i\sqrt{7}$ (C) $-\frac{5}{2} - (\sqrt{2}/2)i$ (D) $\frac{3}{13} + \frac{2}{13}i$
8. (A) $x = -1, y = 4$ (B) $z = 2 + i$

1-4 Exercises

For each number in Problems 1–12, find (A) real part, (B) imaginary part, and (C) conjugate.

1. $2 - 9i$ **2.** $-6i + 4$ **3.** $-\frac{3}{2} + \frac{5}{6}i$

4. $4.2 - 9.7i$ **5.** $6.5 + 2.1i$ **6.** $\frac{3}{5} + \frac{4}{5}i$

7. $i\pi$ **8.** 6π **9.** 4π

10. $-2\pi i$ **11.** $-5 + i\sqrt{2}$ **12.** $4 - i\sqrt{7}$

In Problems 13–38, perform the indicated operations and write each answer in standard form.

13. $(3 + 5i) + (2 + 4i)$ **14.** $(4 + i) + (5 + 3i)$

15. $(8 - 3i) + (-5 + 6i)$ **16.** $(-1 + 2i) + (4 - 7i)$

17. $(9 + 5i) - (6 + 2i)$ **18.** $(3 + 7i) - (2 + 5i)$

19. $(3 - 4i) - (-5 + 6i)$ **20.** $(-4 - 2i) - (1 + i)$

21. $2 + (3i + 5)$ **22.** $(2i + 7) - 4i$

23. $(2i)(4i)$ **24.** $(3i)(5i)$ **25.** $-2i(4 - 6i)$

26. $(-4i)(2 - 3i)$ **27.** $(1 + 2i)(3 - 4i)$

28. $(2 - i)(-5 + 6i)$

29. $(3 - i)(4 + i)$

30. $(5 + 2i)(4 - 3i)$

31. $(2 + 9i)(2 - 9i)$

32. $(3 + 8i)(3 - 8i)$

33. $\dfrac{1}{2 + 4i}$

34. $\dfrac{i}{3 + i}$

35. $\dfrac{4 + 3i}{1 + 2i}$

36. $\dfrac{3 - 5i}{2 - i}$

37. $\dfrac{7 + i}{2 + i}$

38. $\dfrac{-5 + 10i}{3 + 4i}$

In Problems 39–46, evaluate and express results in standard form.

39. $\sqrt{2}\sqrt{8}$

40. $\sqrt{3}\sqrt{12}$

41. $\sqrt{2}\sqrt{-8}$

42. $\sqrt{-3}\sqrt{12}$

43. $\sqrt{-2}\sqrt{8}$

44. $\sqrt{3}\sqrt{-12}$

45. $\sqrt{-2}\sqrt{-8}$

46. $\sqrt{-3}\sqrt{-12}$

In Problems 47–56, convert imaginary numbers to standard form, perform the indicated operations, and express answers in standard form.

47. $(2 - \sqrt{-4}) + (5 - \sqrt{-9})$

48. $(3 - \sqrt{-4}) + (-8 + \sqrt{-25})$

49. $(9 - \sqrt{-9}) - (12 - \sqrt{-25})$

50. $(-2 - \sqrt{-36}) - (4 + \sqrt{-49})$

51. $(3 - \sqrt{-4})(-2 + \sqrt{-49})$

52. $(2 - \sqrt{-1})(5 + \sqrt{-9})$

53. $\dfrac{5 - \sqrt{-4}}{7}$

54. $\dfrac{6 - \sqrt{-64}}{2}$

55. $\dfrac{1}{2 - \sqrt{-9}}$

56. $\dfrac{1}{3 - \sqrt{-16}}$

Write Problems 57–62 in standard form.

57. $\dfrac{2}{5i}$

58. $\dfrac{1}{3i}$

59. $\dfrac{1 + 3i}{2i}$

60. $\dfrac{2 - i}{3i}$

61. $(2 - 3i)^2 - 2(2 - 3i) + 9$

62. $(2 - i)^2 + 3(2 - i) - 5$

63. Evaluate $x^2 - 2x + 2$ for $x = 1 - i$.

64. Evaluate $x^2 - 2x + 2$ for $x = 1 + i$.

In Problems 65–68, for what real values of x does each expression represent an imaginary number?

65. $\sqrt{3 - x}$

66. $\sqrt{5 + x}$

67. $\sqrt{2 - 3x}$

68. $\sqrt{3 + 2x}$

Use a calculator to compute Problems 69–72. Write in standard form a + bi, where a and b are computed to three significant digits.

69. $(3.17 - 4.08i)(7.14 + 2.76i)$

70. $(6.12 + 4.92i)(1.82 - 5.05i)$

71. $\dfrac{8.14 + 2.63i}{3.04 + 6.27i}$

72. $\dfrac{7.66 + 3.33i}{4.72 - 2.68i}$

In Problems 73–76, solve for x and y.

73. $(2x - 1) + (3y + 2)i = 5 - 4i$

74. $3x + (y - 2)i = (5 - 2x) + (3y - 8)i$

75. $\dfrac{(1 + x) + (y - 2)i}{1 + i} = 2 - i$

76. $\dfrac{(2 + x) + (y + 3)i}{1 - i} = -3 + i$

In Problems 77–80, solve for z. Express answers in standard form.

77. $(2 + i)z + i = 4i$

78. $(3 - i)z + 2 = i$

79. $3iz + (2 - 4i) = (1 + 2i)z - 3i$

80. $(2 - i)z + (1 - 4i) = (-1 + 3i)z + (4 + 2i)$

81. Explain what is wrong with the following "proof" that $-1 = 1$:

$$-1 = i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

82. Explain what is wrong with the following "proof" that $1/i = i$. What is the correct value of $1/i$?

$$\frac{1}{i} = \frac{1}{\sqrt{-1}} = \frac{\sqrt{1}}{\sqrt{-1}} = \sqrt{\frac{1}{-1}} = \sqrt{-1} = i$$

In Problems 83–86, perform the indicated operations, and write each answer in standard form.

83. $(a + bi)(a - bi)$

84. $(u - vi)(u + vi)$

85. $(a + bi)(c + di)$

86. $\dfrac{a + bi}{c + di}$

87. Show that $i^{4k} = 1$, k a natural number

88. Show that $i^{4k+1} = i$, k a natural number

Supply the reasons in the proofs for the theorems stated in Problems 89 and 90.

89. *Theorem:* The complex numbers are commutative under addition.

Proof: Let $a + bi$ and $c + di$ be two arbitrary complex numbers; then:

Statement

1. $(a + bi) + (c + di) = (a + c) + (b + d)i$

2. $= (c + a) + (d + b)i$

3. $= (c + di) + (a + bi)$

Reason

1.

2.

3.

90. *Theorem:* The complex numbers are commutative under multiplication.

Proof: Let $a + bi$ and $c + di$ be two arbitrary complex numbers; then:

Statement

1. $(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$

2. $= (ca - db) + (da + cb)i$

3. $= (c + di)(a + bi)$

Reason

1.

2.

3.

Letters z and w are often used as complex variables, where $z = x + yi$, $w = u + vi$, and x, y, u, v are real numbers. The conjugates of z and w, denoted by $\bar{z}$ and $\bar{w}$, respectively, are given by $\bar{z} = x - yi$ and $\bar{w} = u - vi$. In Problems 91–98, express each property of conjugates verbally and then prove the property.

91. $z\bar{z}$ is a real number.

92. $z + \bar{z}$ is a real number.

93. $\bar{z} = z$ if and only if z is real.

94. $\bar{\bar{z}} = z$

95. $\overline{z + w} = \bar{z} + \bar{w}$

96. $\overline{z - w} = \bar{z} - \bar{w}$

97. $\overline{zw} = \bar{z} \cdot \bar{w}$

98. $\overline{z/w} = \bar{z}/\bar{w}$

1-5 Quadratic Equations and Applications

- Using Factoring to Solve Quadratic Equations
- Using the Square Root Property to Solve Quadratic Equations
- Using Completing the Square to Solve Quadratic Equations
- Using the Quadratic Formula to Solve Quadratic Equations
- Solving Applications Involving Quadratic Equations

The next class of equations we consider are the second-degree polynomial equations in one variable, called *quadratic equations.*

DEFINITION 1 Quadratic Equation

A **quadratic equation** in one variable is any equation that can be written in the form

$$ax^2 + bx + c = 0 \qquad a \neq 0 \qquad \textbf{Standard Form}$$

where x is a variable and a, b, and c are constants.

Now that we have discussed the complex number system, we will use complex numbers when solving equations. Recall that a solution of an equation is also called a *root* of the equation. A real number solution of an equation is called a **real root,** and an imaginary number solution is called an **imaginary root.** In this section we develop methods for finding all real and imaginary roots of a quadratic equation.

Using Factoring to Solve Quadratic Equations

If $ax^2 + bx + c$ can be written as the product of two first-degree factors, then the quadratic equation can be quickly and easily solved. The method of solution by factoring rests on the zero property of complex numbers, which is a generalization of the zero property of real numbers introduced in Section R-1.

ZERO PROPERTY

If m and n are complex numbers, then

$$m \cdot n = 0 \qquad \text{if and only if} \qquad m = 0 \text{ or } n = 0 \text{ (or both)}$$

EXAMPLE 1 **Solving Quadratic Equations by Factoring**

Solve by factoring:

(A) $6x^2 - 19x - 7 = 0$ (B) $x^2 - 6x + 5 = -4$ (C) $2x^2 = 3x$

SOLUTIONS

(A) $6x^2 - 19x - 7 = 0$ Factor left side.

$(2x - 7)(3x + 1) = 0$ Apply the zero property.

$2x - 7 = 0$ or $3x + 1 = 0$

$x = \frac{7}{2}$ $\qquad$ $x = -\frac{1}{3}$

The solution set is $\{-\frac{1}{3}, \frac{7}{2}\}$.

(B) $x^2 - 6x + 5 = -4$ Write in standard form.

$x^2 - 6x + 9 = 0$ Factor left side.

$(x - 3)^2 = 0$ Apply the zero property.

$x = 3$

The solution set is $\{3\}$. The equation has one root, 3. But since it came from two factors, we call 3 a **double root.**

(C) $2x^2 = 3x$ Write in standard form.

$2x^2 - 3x = 0$ Factor left side.

$x(2x - 3) = 0$ Apply the zero property.

$x = 0$ or $2x - 3 = 0$

$x = \frac{3}{2}$

Solution set: $\{0, \frac{3}{2}\}$

MATCHED PROBLEM 1

Solve by factoring:

(A) $3x^2 + 7x - 20 = 0$ (B) $4x^2 + 12x + 9 = 0$ (C) $4x^2 = 5x$

››› CAUTION ›››

1. One side of an equation must be 0 before the zero property can be applied. Thus,

$$x^2 - 6x + 5 = -4$$
$$(x - 1)(x - 5) = -4$$

does not imply that $x - 1 = -4$ or $x - 5 = -4$. See Example 1B for the correct solution of this equation.

2. The equations

$$2x^2 = 3x \qquad \text{and} \qquad 2x = 3$$

are not equivalent. The first has solution set $\{0, \frac{3}{2}\}$, while the second has solution set $\{\frac{3}{2}\}$. The root $x = 0$ is lost when each member of the first equation is divided by the variable x. See Example 1C for the correct solution of this equation.

Do not divide both sides of an equation by an expression containing the variable for which you are solving. You may be dividing by 0.

Using the Square Root Property to Solve Quadratic Equations

We now turn our attention to quadratic equations that do not have the first-degree term—that is, equations of the special form

$$ax^2 + c = 0 \qquad a \neq 0$$

The method of solution of this special form makes direct use of the square root property:

SQUARE ROOT PROPERTY

If $A^2 = C$, then $A = \pm\sqrt{C}$.

EXPLORE-DISCUSS 1

Determine if each of the following pairs of equations is equivalent or not. Explain your answer.

(A) $x^2 = 4$ and $x = |2|$

(B) $x^2 = 4$ and $x = -2$

(C) $x = \sqrt{4}$ and $x = 2$

(D) $x = \sqrt{4}$ and $x = -2$

The use of the square root property is illustrated in Example 2.

EXAMPLE 2 Using the Square Root Property

Solve using the square root property:

(A) $2x^2 - 3 = 0$ (B) $3x^2 + 27 = 0$ (C) $(x + \frac{1}{2})^2 = \frac{5}{4}$

SOLUTIONS

(A) $2x^2 - 3 = 0$ Solve for x^2.

$x^2 = \frac{3}{2}$ Apply the square root property.

$x = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$ Solution set: $\left\{\frac{-\sqrt{6}}{2}, \frac{\sqrt{6}}{2}\right\}$

(B) $3x^2 + 27 = 0$ — Solve for x^2.

$x^2 = -9$ — Apply the square root property.

$x = \pm\sqrt{-9} = \pm 3i$ — Solution set: $\{-3i, 3i\}$

(C) $(x + \frac{1}{2})^2 = \frac{5}{4}$ — Apply the square root property.

$x + \frac{1}{2} = \pm\sqrt{\frac{5}{4}}$ — Subtract $\frac{1}{2}$ from both sides, and simplify $\sqrt{\frac{5}{4}}$.

$$x = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

MATCHED PROBLEM 2

Solve using the square root property:

(A) $3x^2 - 5 = 0$ (B) $2x^2 + 8 = 0$ (C) $(x + \frac{1}{3})^2 = \frac{2}{9}$

Note: It is common practice to represent solutions of quadratic equations informally by the last equation (Example 2C) rather than by writing a solution set using set notation (Examples 2A and 2B). From now on, we will follow this practice unless a particular emphasis is desired.

EXPLORE-DISCUSS 2

Replace ? in each of the following with a number that makes the equation valid.

(A) $(x + 1)^2 = x^2 + 2x + ?$ **(B)** $(x + 2)^2 = x^2 + 4x + ?$

(C) $(x + 3)^2 = x^2 + 6x + ?$ **(D)** $(x + 4)^2 = x^2 + 8x + ?$

Replace ? in each of the following with a number that makes the trinomial a perfect square.

(E) $x^2 + 10x + ?$ **(F)** $x^2 + 12x + ?$ **(G)** $x^2 + bx + ?$

Using Completing the Square to Solve Quadratic Equations

The methods of square root and factoring are generally fast when they apply; however, there are equations, such as $x^2 + 6x - 2 = 0$ (see Example 4A), that cannot be solved directly by these methods. A more general procedure must be developed to take care of this type of equation—for example, the method of completing the square.* This method is based on the process of transforming the standard quadratic equation

*We will find many other uses for this important method.

$$ax^2 + bx + c = 0$$

into the form

$$(x + A)^2 = B$$

where A and B are constants. The last equation can easily be solved by using the square root property. But how do we transform the first equation into the second? The following brief discussion provides the key to the process.

What number must be added to $x^2 + bx$ so that the result is the square of a first-degree polynomial? There is a simple mechanical rule for finding this number, based on the square of the following binomials:

$$(x + m)^2 = x^2 + 2mx + m^2$$
$$(x - m)^2 = x^2 - 2mx + m^2$$

In either case, we see that the third term on the right is the square of one-half of the coefficient of x in the second term on the right. This observation leads directly to the rule for completing the square.

› COMPLETING THE SQUARE

To complete the square of a quadratic expression of the form $x^2 + bx$, add the square of one-half the coefficient of x; that is, add $(b/2)^2$. Thus,

$$x^2 + bx \qquad\qquad x^2 + 5x$$

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 \qquad\qquad x^2 + 5x + \left(\frac{5}{2}\right)^2 = \left(x + \frac{5}{2}\right)^2$$

To complete the square in a quadratic equation, add $(b/2)^2$ to both sides of the equation. See Example 4.

EXAMPLE 3 Completing the Square

Complete the square for each of the following:

(A) $x^2 - 3x$ (B) $x^2 - bx$

SOLUTIONS

(A) $x^2 - 3x$

$$x^2 - 3x + \tfrac{9}{4} = (x - \tfrac{3}{2})^2$$ Add $\left(\frac{-3}{2}\right)^2$; that is, $\frac{9}{4}$ and factor.

(B) $x^2 - bx$

$$x^2 - bx + \frac{b^2}{4} = \left(x - \frac{b}{2}\right)^2$$ Add $\left(\frac{-b}{2}\right)^2$; that is, $\frac{b^2}{4}$ and factor.

MATCHED PROBLEM 3

Complete the square for each of the following:

(A) $x^2 - 5x$ (B) $x^2 + mx$

You should note that the rule for completing the square applies only if the coefficient of the second-degree term is 1. This causes little trouble, however, as you will see. We now solve two equations by the method of completing the square.

EXAMPLE 4 Solution by Completing the Square

Solve by completing the square:

(A) $x^2 + 6x - 2 = 0$ (B) $2x^2 - 4x + 3 = 0$

SOLUTIONS

(A)

$x^2 + 6x - 2 = 0$	Add 2 to both sides to obtain the form $x^2 + bx$ on the left side.
$x^2 + 6x = 2$	Complete the square on the left side and add $(\frac{b}{2})^2 = (\frac{6}{2})^2 = 9$ to both sides.
$x^2 + 6x + 9 = 2 + 9$	Factor the left side.
$(x + 3)^2 = 11$	Use the square root property.
$x + 3 = \pm\sqrt{11}$	Add -3 to both sides.
$x = -3 \pm \sqrt{11}$	

(B)

$2x^2 - 4x + 3 = 0$	Make the leading coefficient 1 by dividing by 2.
$x^2 - 2x + \frac{3}{2} = 0$	Subtract $-\frac{3}{2}$ from both sides.
$x^2 - 2x = -\frac{3}{2}$	Complete the square on the left side and add $(\frac{b}{2})^2 = (\frac{-2}{2})^2 = 1$ to both sides.
$x^2 - 2x + 1 = -\frac{3}{2} + 1$	Factor the left side.
$(x - 1)^2 = -\frac{1}{2}$	Use the square root property.
$x - 1 = \pm\sqrt{-\frac{1}{2}}$	Add 1 to both sides and simplify $\sqrt{-\frac{1}{2}}$.
$x = 1 \pm i\sqrt{\frac{1}{2}}$	
$= 1 \pm \frac{\sqrt{2}}{2}i$	Answer in $a + bi$ form.

MATCHED PROBLEM 4

Solve by completing the square:

(A) $x^2 + 8x - 3 = 0$ (B) $3x^2 - 12x + 13 = 0$

Using the Quadratic Formula to Solve Quadratic Equations

Now consider the general quadratic equation with unspecified coefficients:

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

$$ax^2 + bx + c = 0$$ Make the leading coefficient 1 by dividing by a.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$ Subtract $\frac{c}{a}$ from both sides.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$ Complete the square on the left side and add $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$ to both sides.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$ Factor the left side and combine terms on the right side.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$ Use the square root property.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$ Add $-\frac{b}{2a}$ to both sides and simplify $\sqrt{\frac{b^2 - 4ac}{4a^2}}$ (see Problem 73 in Exercises 1-5).

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$ Combine terms on the right side.

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We have thus derived the well-known and widely used **quadratic formula:**

THEOREM 1 Quadratic Formula

If $ax^2 + bx + c = 0$, $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula should be memorized and used to solve quadratic equations when other methods fail or are more difficult to apply.

EXAMPLE 5

Using the Quadratic Formula

Solve $2x + \frac{3}{2} = x^2$ by use of the quadratic formula. Leave the answer in simplest radical form.

SOLUTION

$$2x + \tfrac{3}{2} = x^2 \qquad \text{Multiply both sides by 2.}$$

$$4x + 3 = 2x^2 \qquad \text{Write in standard form.}$$

$$2x^2 - 4x - 3 = 0 \qquad \text{Identify } a, b, \text{ and } c \text{ and use the quadratic formula: } a = 2,\ b = -4,\ c = -3$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{4 \pm \sqrt{40}}{4} = \frac{4 \pm 2\sqrt{10}}{4} = \frac{2 \pm \sqrt{10}}{2}$$

››› CAUTION ›››

1. $-4^2 \neq (-4)^2$ — $-4^2 = -16$ and $(-4)^2 = 16$

2. $2 + \dfrac{\sqrt{10}}{2} \neq \dfrac{2 + \sqrt{10}}{2}$ — $2 + \dfrac{\sqrt{10}}{2} = \dfrac{4 + \sqrt{10}}{2}$

3. $\dfrac{\not{4} \pm 2\sqrt{10}}{\not{4}} \neq \pm 2\sqrt{10}$ — $\dfrac{4 \pm 2\sqrt{10}}{4} = \dfrac{2(2 \pm \sqrt{10})}{4} = \dfrac{2 \pm \sqrt{10}}{2}$

MATCHED PROBLEM 5

Solve $x^2 - \frac{5}{2} = -3x$ by use of the quadratic formula. Leave the answer in simplest radical form.

We conclude this part of the discussion by noting that $b^2 - 4ac$ in the quadratic formula is called the **discriminant** and gives us useful information about the corresponding roots as shown in Table 1.

Table 1 Discriminant and Roots

Discriminant $b^2 - 4ac$	Roots of $ax^2 + bx + c = 0$ a, b, and c real numbers, $a \neq 0$
Positive	Two distinct real roots
0	One real root (a double root)
Negative	Two imaginary roots, one the conjugate of the other

EXAMPLE 6 Using the Discriminant

Find the number of real roots of each quadratic equation.

(A) $2x^2 - 4x + 1 = 0$ (B) $2x^2 - 4x + 2 = 0$ (C) $2x^2 - 4x + 3 = 0$

SOLUTIONS

(A) $b^2 - 4ac = (-4)^2 - 4(2)(1) = 8 > 0$; two real roots

(B) $b^2 - 4ac = (-4)^2 - 4(2)(2) = 0$; one real (double) root

(C) $b^2 - 4ac = (-4)^2 - 4(2)(3) = -8 < 0$; no real roots (two imaginary roots)

MATCHED PROBLEM 6

Find the number of real roots of each quadratic equation.

(A) $3x^2 - 6x + 5 = 0$ (B) $3x^2 - 6x + 1 = 0$ (C) $3x^2 - 6x + 3 = 0$

Solving Applications Involving Quadratic Equations

We now consider several applications that involve finding solutions to quadratic equations. At this point, you should review the strategy for solving word problems presented in Section 1-1.

STRATEGY FOR SOLVING WORD PROBLEMS

1. Read the problem carefully—several times if necessary—that is, until you understand the problem, know what is to be found, and know what is given.
2. Let one of the unknown quantities be represented by a variable, say x, and try to represent all other unknown quantities in terms of x. This is an important step and must be done carefully.
3. If appropriate, draw figures or diagrams and label known and unknown parts.
4. Look for formulas connecting the known quantities to the unknown quantities.
5. Form an equation relating the unknown quantities to the known quantities.
6. Solve the equation and write answers to *all* questions asked in the problem.
7. Check and interpret all solutions in terms of the original problem—not just the equation found in step 5—since a mistake may have been made in setting up the equation in step 5.

EXAMPLE 7 Setting Up and Solving a Word Problem

The sum of a number and its reciprocal is $\frac{13}{6}$. Find all such numbers.

SOLUTION

Let x = the number; then:

$$x + \frac{1}{x} = \frac{13}{6}$$ Multiply both sides by $6x$. [*Note:* $x \neq 0$.]

$$(6x)x + (6x)\frac{1}{x} = (6x)\frac{13}{6}$$

$$6x^2 + 6 = 13x$$ Subtract $13x$ from both sides.

$$6x^2 - 13x + 6 = 0$$ Factor the left side.

$$(2x - 3)(3x - 2) = 0$$ Use the zero property.

$$2x - 3 = 0 \quad \text{or} \quad 3x - 2 = 0$$ Solve each equation for x.

$$x = \tfrac{3}{2} \qquad x = \tfrac{2}{3}$$

Thus, two such numbers are $\frac{3}{2}$ and $\frac{2}{3}$.

CHECK $\frac{3}{2} + \frac{2}{3} = \frac{13}{6}$ $\frac{2}{3} + \frac{3}{2} = \frac{13}{6}$

MATCHED PROBLEM 7

The sum of two numbers is 23 and their product is 132. Find the two numbers. [*Hint:* If one number is x, then the other number is $23 - x$.]

EXAMPLE 8 A Distance–Rate–Time Problem

An excursion boat takes 1.6 hours longer to go 36 miles up a river than to return. If the rate of the current is 4 miles per hour, what is the rate of the boat in still water?

SOLUTION

Let

x = Rate of boat in still water

$x + 4$ = Rate downstream

$x - 4$ = Rate upstream

$$\begin{aligned}
\begin{pmatrix}\text{Time}\\ \text{upstream}\end{pmatrix} - \begin{pmatrix}\text{Time}\\ \text{downstream}\end{pmatrix} &= 1.6 && T = \frac{D}{R},\ x \neq 4,\ x \neq -4\\
\frac{36}{x-4} - \frac{36}{x+4} &= 1.6 && \text{Multiply both sides by } (x-4)(x+4) \text{, the LCD.}\\
36(x+4) - 36(x-4) &= 1.6(x-4)(x+4) && \text{Remove parentheses.}\\
36x + 144 - 36x + 144 &= 1.6x^2 - 25.6 && \text{Combine like terms and isolate } 1.6x^2 \text{ on one side of the equation.}\\
1.6x^2 &= 313.6 && \text{Divide both sides by 1.6.}\\
x^2 &= 196 && \text{Use the square root property.}\\
x &= \sqrt{196} = 14
\end{aligned}$$

The rate in still water is 14 miles per hour.

[*Note:* $-\sqrt{196} = -14$ must be discarded, since it doesn't make sense in the problem to have a negative rate.]

CHECK

$$\text{Time upstream} = \frac{D}{R} = \frac{36}{14-4} = 3.6$$

$$\text{Time downstream} = \frac{D}{R} = \frac{36}{14+4} = 2$$

$$\overline{1.6} \qquad \text{Difference of times}$$

MATCHED PROBLEM 8

Two boats travel at right angles to each other after leaving a dock at the same time. One hour later they are 25 miles apart. If one boat travels 5 miles per hour faster than the other, what is the rate of each? [*Hint:* Use the Pythagorean theorem,* remembering that distance equals rate times time.]

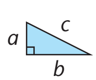
EXAMPLE 9 A Quantity–Rate–Time Problem

A payroll can be completed in 4 hours by two computers working simultaneously. How many hours are required for each computer to complete the payroll alone if the older model requires 3 hours longer than the newer model? Compute answers to two decimal places.

SOLUTION

Let

$$\begin{aligned}
x &= \text{Time for new model to complete the payroll alone}\\
x + 3 &= \text{Time for old model to complete the payroll alone}\\
4 &= \text{Time for both computers to complete the payroll together}
\end{aligned}$$



**Pythagorean theorem:* A triangle is a right triangle if and only if the square of the length of the longest side is equal to the sum of the squares of the lengths of the two shorter sides: $c^2 = a^2 + b^2$.

Then,

$$\frac{1}{x} = \text{Rate for new model} \qquad \text{Completes } \frac{1}{x} \text{ of the payroll per hour}$$

$$\frac{1}{x+3} = \text{Rate for old model} \qquad \text{Completes } \frac{1}{x+3} \text{ of the payroll per hour}$$

$$\begin{pmatrix}\text{Part of job}\\ \text{completed by}\\ \text{new model in}\\ \text{4 hours}\end{pmatrix} + \begin{pmatrix}\text{Part of job}\\ \text{completed by}\\ \text{old model in}\\ \text{4 hours}\end{pmatrix} = \text{1 whole job}$$

$$\frac{1}{x}(4) + \frac{1}{x+3}(4) = 1 \qquad x \neq 0, x \neq -3$$

$$\frac{4}{x} + \frac{4}{x+3} = 1 \qquad \text{Multiply both sides by } x(x+3).$$

$$4(x+3) + 4x = x(x+3) \qquad \text{Remove parentheses.}$$

$$4x + 12 + 4x = x^2 + 3x \qquad \text{Write in standard form.}$$

$$x^2 - 5x - 12 = 0 \qquad \text{Use the quadratic formula.}$$

$$x = \frac{5 \pm \sqrt{73}}{2}$$

$$x = \frac{5 + \sqrt{73}}{2} \approx 6.77 \qquad \frac{5 - \sqrt{73}}{2} \approx -1.77 \text{ is discarded since } x \text{ cannot be negative.}$$

$$x + 3 = 9.77$$

The new model would complete the payroll in 6.77 hours working alone, and the old model would complete the payroll in 9.77 hours working alone.

CHECK

$$\frac{1}{6.77}(4) + \frac{1}{9.77}(4) \stackrel{?}{=} 1$$

$$1.000\ 259 \stackrel{\checkmark}{\approx} 1$$

Note: We do not expect the check to be exact, since we rounded the answers to two decimal places. An exact check would be produced by using $x = (5 + \sqrt{73})/2$. The latter is left to the reader.

MATCHED PROBLEM 9

Two technicians can complete a mailing in 3 hours when working together. Alone, one can complete the mailing 2 hours faster than the other. How long will it take each person to complete the mailing alone? Compute the answers to two decimal places.

In Example 10, we introduce some concepts from economics that will be used throughout this book. The quantity of a product that people are willing to buy during some period of time is called the **demand** for that product. The price p of a product and the demand q for that product are often related by a **price–demand equation** of the following form:

$$q = a - bp \qquad \text{q is the number of items that can be sold at \$$p$ per item.}$$

The constants a and b in a price–demand equation are usually determined by using historical data and statistical analysis.

The amount of money received from the sale of q items at \$$p$ per item is called the **revenue** and is given by

$$\begin{aligned} R &= (\text{Number of items sold}) \times (\text{Price per item}) \\ &= qp = q(a - bp) \end{aligned}$$

EXAMPLE 10 Price and Demand

The daily price–demand equation for whole milk in a chain of supermarkets is

$$q = 5{,}600 - 800p$$

where p is the price per gallon and q is the number of gallons sold per day. Find the price(s) that will produce a revenue of \$9,500. Round answer(s) to two decimal places.

SOLUTION

The revenue equation is

$$\begin{aligned} R &= qp = (5{,}600 - 800p)p \\ &= 5{,}600p - 800p^2 \end{aligned}$$

To produce a revenue of \$9,500, p must satisfy

$$\begin{aligned} 5{,}600p - 800p^2 &= 9{,}500 && \text{Subtract 9,500 from both sides.} \\ -9{,}500 + 5{,}600p - 800p^2 &= 0 && \text{Divide both sides by } -800. \\ p^2 - 7p + 11.875 &= 0 && \text{Use the quadratic formula with } a = 1, \\ & && b = -7, \text{ and } c = 11.875. \\ p &= \frac{7 \pm \sqrt{1.5}}{2} \\ &= 2.89, 4.11 \end{aligned}$$

Selling whole milk for either \$2.89 per gallon or \$4.11 per gallon will produce a revenue of \$9,500.

MATCHED PROBLEM 10

Refer to Example 10. Find the prices that will produce a revenue of

(A) \$9,300 (B) \$9,800

ANSWERS TO MATCHED PROBLEMS

1. (A) $x = -4, \frac{5}{3}$ (B) $x = -\frac{3}{2}$ (a double root) (C) $x = 0, \frac{5}{4}$
2. (A) $x = \pm\sqrt{\frac{5}{3}}$ or $\pm\sqrt{15}/3$ (B) $x = \pm 2i$ (C) $x = (-1 \pm \sqrt{2})/3$
3. (A) $x^2 - 5x + \frac{25}{4} = (x - \frac{5}{2})^2$ (B) $x^2 + mx + (m^2/4) = [x + (m/2)]^2$
4. (A) $x = -4 \pm \sqrt{19}$ (B) $x = (6 \pm i\sqrt{3})/3$ or $2 \pm (\sqrt{3}/3)i$
5. $x = (-3 \pm \sqrt{19})/2$ **6.** (A) No real roots (two imaginary roots) (B) Two real roots (C) One real (double) root **7.** 11 and 12 **8.** 15 and 20 miles per hour
9. 5.16 and 7.16 hours **10.** (A) \$2.71, \$4.29 (B) \$3.50

1-5 Exercises

Leave all answers involving radicals in simplified radical form unless otherwise stated.

In Problems 1–6, solve by factoring.

1. $2x^2 = 8x$ **2.** $2y^2 + 5y = 3$

3. $4t^2 + 9 = 12t$ **4.** $3s^2 = -6s$

5. $3w^2 + 13w = 10$ **6.** $16x^2 + 9 = 24x$

In Problems 7–18, solve by using the square root property.

7. $m^2 - 25 = 0$ **8.** $n^2 + 16 = 0$

9. $c^2 + 9 = 0$ **10.** $d^2 - 36 = 0$

11. $4y^2 + 9 = 0$ **12.** $9x^2 - 25 = 0$

13. $25z^2 - 32 = 0$ **14.** $16w^2 + 27 = 0$

15. $(s + 1)^2 = 5$ **16.** $(t - 2)^2 = -3$

17. $(n - 3)^2 = -4$ **18.** $(m + 4)^2 = 1$

In Problems 19–26, use the discriminant to determine the number of real roots of each equation and then solve each equation using the quadratic formula.

19. $x^2 - 2x - 1 = 0$ **20.** $y^2 - 4y + 7 = 0$

21. $x^2 - 2x + 3 = 0$ **22.** $y^2 - 4y + 1 = 0$

23. $2t^2 + 8 = 6t$ **24.** $9s^2 + 2 = 12s$

25. $2t^2 + 1 = 6t$ **26.** $9s^2 + 7 = 12s$

In Problems 27–34, solve by completing the square.

27. $x^2 - 4x - 1 = 0$ **28.** $y^2 + 4y - 3 = 0$

29. $2r^2 + 10r + 11 = 0$ **30.** $2s^2 - 6s + 7 = 0$

31. $4u^2 + 8u + 15 = 0$ **32.** $4v^2 + 16v + 23 = 0$

33. $3w^2 + 4w + 3 = 0$ **34.** $3z^2 - 8z + 1 = 0$

In Problems 35–52, solve by any method.

35. $12x^2 + 7x = 10$ **36.** $9x^2 + 9x = 4$

37. $(2y - 3)^2 = 5$ **38.** $(3m + 2)^2 = -4$

39. $x^2 = 3x + 1$ **40.** $x^2 + 2x = 2$

41. $7n^2 = -4n$ **42.** $8u^2 + 3u = 0$

43. $1 + \frac{8}{x^2} = \frac{4}{x}$ **44.** $\frac{2}{u} = \frac{3}{u^2} + 1$

45. $\frac{24}{10 + m} + 1 = \frac{24}{10 - m}$

46. $\frac{1.2}{y - 1} + \frac{1.2}{y} = 1$ **47.** $\frac{2}{x - 2} = \frac{4}{x - 3} - \frac{1}{x + 1}$

48. $\frac{3}{x - 1} - \frac{2}{x + 3} = \frac{4}{x - 2}$

49. $\frac{x + 2}{x + 3} - \frac{x^2}{x^2 - 9} = 1 - \frac{x - 1}{3 - x}$

50. $\frac{11}{x^2 - 4} + \frac{x + 3}{2 - x} = \frac{2x - 3}{x + 2}$

51. $|3u - 2| = u^2$ **52.** $|12 + 7x| = x^2$

In Problems 53–56, solve for the indicated variable in terms of the other variables. Use positive square roots only.

53. $s = \frac{1}{2}gt^2$ for t

54. $a^2 + b^2 = c^2$ for a

55. $P = EI - RI^2$ for I

56. $A = P(1 + r)^2$ for r

Solve Problems 57–60 to three significant digits.

57. $2.07x^2 - 3.79x + 1.34 = 0$

58. $0.610x^2 - 4.28x + 2.93 = 0$

59. $4.83x^2 + 2.04x - 3.18 = 0$

60. $5.13x^2 + 7.27x - 4.32 = 0$

61. Consider the quadratic equation

$$x^2 + 4x + c = 0$$

where c is a real number. Discuss the relationship between the values of c and the three types of roots listed in Table 1.

62. Consider the quadratic equation

$$x^2 - 2x + c = 0$$

where c is a real number. Discuss the relationship between the values of c and the three types of roots listed in Table 1.

Solve Problems 63–66 and leave answers in simplified radical form (i is the imaginary unit).

63. $\sqrt{3}x^2 = 8\sqrt{2}x - 4\sqrt{3}$

64. $2\sqrt{2}x + \sqrt{3} = \sqrt{3}x^2$

65. $x^2 + 2ix = 3$

66. $x^2 = 2ix - 3$

In Problems 67 and 68, find all solutions.

67. $x^3 - 1 = 0$

68. $x^4 - 1 = 0$

69. Can a quadratic equation with rational coefficients have one rational root and one irrational root? Explain.

70. Can a quadratic equation with real coefficients have one real root and one imaginary root? Explain.

71. Show that if r_1 and r_2 are the two roots of $ax^2 + bx + c = 0$, then $r_1r_2 = c/a$.

72. For r_1 and r_2 in Problem 71, show that $r_1 + r_2 = -b/a$.

73. In one stage of the derivation of the quadratic formula, we replaced the expression

$$\pm\sqrt{(b^2 - 4ac)/4a^2}$$

with

$$\pm\sqrt{b^2 - 4ac}/2a$$

What justifies using $2a$ in place of $|2a|$?

74. Find the error in the following "proof" that two arbitrary numbers are equal to each other: Let a and b be arbitrary numbers such that $a \neq b$. Then

$$(a - b)^2 = a^2 - 2ab + b^2 = b^2 - 2ab + a^2$$
$$(a - b)^2 = (b - a)^2$$
$$a - b = b - a$$
$$2a = 2b$$
$$a = b$$

APPLICATIONS

75. NUMBERS Find two numbers such that their sum is 21 and their product is 104.

76. NUMBERS Find all numbers with the property that when the number is added to itself the sum is the same as when the number is multiplied by itself.

77. NUMBERS Find two consecutive positive even integers whose product is 168.

78. NUMBERS The sum of a number and its reciprocal is $\frac{10}{3}$. Find the number.

79. GEOMETRY If the length and width of a 4- by 2-inch rectangle are each increased by the same amount, the area of the new rectangle will be twice that of the original. What are the dimensions of the new rectangle (to two decimal places)?

80. GEOMETRY Find the base b and height h of a triangle with an area of 2 square feet if its base is 3 feet longer than its height and the formula for area is $A = \frac{1}{2}bh$.

81. PRICE AND DEMAND The daily price–demand equation for hamburgers at a fast-food restaurant is

$$q = 1{,}600 - 200p$$

where q is the number of hamburgers sold daily and p is the price of one hamburger (in dollars). Find the demand and the revenue when the price of a hamburger is \$3.

82. PRICE AND DEMAND The weekly price–demand equation for medium pepperoni pizzas at a fast-food restaurant is

$$q = 8{,}000 - 400p$$

where q is the number of pizzas sold weekly and p is the price of one medium pepperoni pizza (in dollars). Find the demand and the revenue when the price is \$8.

83. PRICE AND DEMAND Refer to Problem 81. Find the price p that will produce each of the following revenues. Round answers to two decimal places.
(A) \$2,800 (B) \$3,200 (C) \$3,400

84. **PRICE AND DEMAND** Refer to Problem 82. Find the price p that will produce each of the following revenues. Round answers to two decimal places.
(A) \$38,000 (B) \$40,000 (C) \$42,000

85. **PUZZLE** Two planes travel at right angles to each other after leaving the same airport at the same time. One hour later they are 260 miles apart. If one travels 140 miles per hour faster than the other, what is the rate of each?

86. **NAVIGATION** A speedboat takes 1 hour longer to go 24 miles up a river than to return. If the boat cruises at 10 miles per hour in still water, what is the rate of the current?

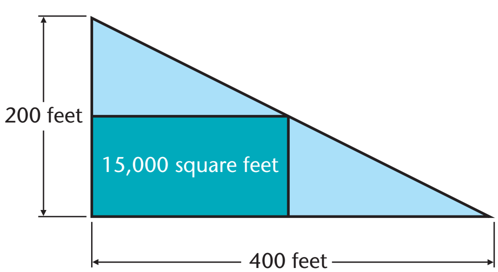
★**87.** **ENGINEERING** One pipe can fill a tank in 5 hours less than another. Together they can fill the tank in 5 hours. How long would it take each alone to fill the tank? Compute the answer to two decimal places.

★★**88.** **ENGINEERING** Two gears rotate so that one completes 1 more revolution per minute than the other. If it takes the smaller gear 1 second less than the larger gear to complete $\frac{1}{5}$ revolution, how many revolutions does each gear make in 1 minute?

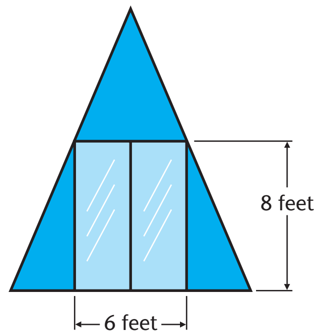
★**89.** **PHYSICS—ENGINEERING** For a car traveling at a speed of v miles per hour, under the best possible conditions the shortest distance d necessary to stop it (including reaction time) is given by the empirical formula $d = 0.044v^2 + 1.1v$, where d is measured in feet. Estimate the speed of a car that requires 165 feet to stop in an emergency.

★**90.** **PHYSICS—ENGINEERING** If a projectile is shot vertically into the air (from the ground) with an initial velocity of 176 feet per second, its distance y (in feet) above the ground t seconds after it is shot is given by $y = 176t - 16t^2$ (neglecting air resistance).
(A) Find the times when y is 0, and interpret the results physically.
(B) Find the times when the projectile is 16 feet off the ground. Compute answers to two decimal places.

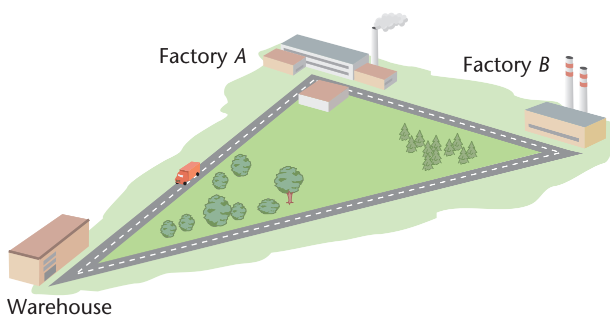
★**91.** **CONSTRUCTION** A developer wants to erect a rectangular building on a triangular-shaped piece of property that is 200 feet wide and 400 feet long (see the figure). Find the dimensions of the building if its cross-sectional area is 15,000 square feet. [*Hint:* Use Euclid's theorem* to find a relationship between the length and width of the building.]



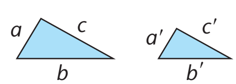
★**92.** **ARCHITECTURE** An architect is designing a small A-frame cottage for a resort area. A cross section of the cottage is an isosceles triangle with an area of 98 square feet. The front wall of the cottage must accommodate a sliding door that is 6 feet wide and 8 feet high (see the figure). Find the width and height of the cross section of the cottage. [*Recall:* The area of a triangle with base b and altitude h is $bh/2$.]



93. **TRANSPORTATION** A delivery truck leaves a warehouse and travels north to factory A. From factory A the truck travels east to factory B and then returns directly to the warehouse (see the figure). The driver recorded the truck's odometer reading at the warehouse at both the beginning and the end of the trip and also at factory B, but forgot to record it at factory A (see the table). The driver does recall that it was farther from the warehouse to factory A than it was from factory A to factory B. Since delivery charges are based on distance from the warehouse, the driver needs to know how far factory A is from the warehouse. Find this distance.



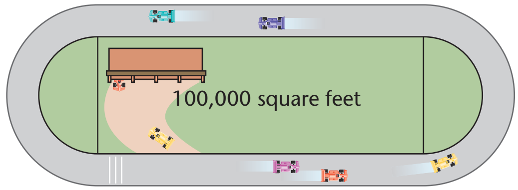
**Euclid's theorem:* If two triangles are similar, their corresponding sides are proportional:



$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

Odometer readings	
Warehouse	52846
Factory A	52???
Factory B	52937
Warehouse	53002

★★**94.** CONSTRUCTION A $\frac{1}{4}$-mile track for racing stock cars consists of two semicircles connected by parallel straightaways (see the figure). In order to provide sufficient room for pit crews, emergency vehicles, and spectator parking, the track must enclose an area of 100,000 square feet. Find the length of the straightaways and the diameter of the semicircles to the nearest foot. [*Recall:* The area A and circumference C of a circle of diameter d are given by $A = \pi d^2/4$ and $c = \pi d$.]



1-6 Additional Equation-Solving Techniques

› Solving Equations Involving Radicals
› Revisiting Equations Involving Absolute Value
› Solving Equations Involving Rational Exponents

In this section we show that raising each side of an equation to the same power can lead to the solutions of the equation. We also show that if changing the variable in an equation transforms the equation into a quadratic equation, then any of the solution techniques discussed in Section 1-5 can be applied to the transformed equation.

› Solving Equations Involving Radicals

In solving an equation involving a radical like

$$x = \sqrt{x + 2}$$

it appears that we can remove the radical by squaring each side and then proceed to solve the resulting quadratic equation. Thus,

$$
\begin{aligned}
x &= \sqrt{x+2} && \text{Square both sides.}\\
x^2 &= (\sqrt{x+2})^2 && \text{Recall that } (\sqrt{a})^2 = a \text{ if } a \ge 0.\\
x^2 &= x + 2 && \text{Subtract } x + 2 \text{ from both sides.}\\
x^2 - x - 2 &= 0 && \text{Factor the left side.}\\
(x - 2)(x + 1) &= 0 && \text{Use the zero property.}\\
x - 2 = 0 \quad &\text{or} \quad x + 1 = 0\\
x = 2 \quad &\text{or} \quad x = -1
\end{aligned}
$$

Now we check these results in the original equation.

Check: $x = 2$	Check: $x = -1$
$x = \sqrt{x + 2}$	$x = \sqrt{x + 2}$
$2 \stackrel{?}{=} \sqrt{2 + 2}$	$-1 \stackrel{?}{=} \sqrt{-1 + 2}$
$2 \stackrel{?}{=} \sqrt{4}$	$-1 \stackrel{?}{=} \sqrt{1}$
$2 \stackrel{\checkmark}{=} 2$	$-1 \neq 1$

Thus, 2 is a solution, but -1 is not. These results are a special case of Theorem 1.

› THEOREM 1 Squaring Operation on Equations

If both sides of an equation are squared, then the solution set of the original equation is a subset of the solution set of the new equation.

Equation	Solution Set
$x = 3$	$\{3\}$
$x^2 = 9$	$\{-3, 3\}$

This theorem provides us with a method of solving some equations involving radicals. It is important to remember that any new equation obtained by raising both members of an equation to the same power may have solutions, called **extraneous solutions,** that are not solutions of the original equation. On the other hand, any solution of the original equation must be among those of the new equation.

Every solution of the new equation must be checked in the original equation to eliminate extraneous solutions.

››› EXPLORE-DISCUSS 1

Squaring both sides of the equations $x = \sqrt{x}$ and $x = -\sqrt{x}$ produces the new equation $x^2 = x$. Find the solutions to the new equation and then check for extraneous solutions in each of the original equations.

››› **CAUTION** ›››

Remember that $\sqrt{9}$ represents the *positive* square root of 9 and $-\sqrt{9}$ represents the *negative* square root of 9. It is correct to use the symbol $\pm$ to combine these two roots when solving an equation:

$$x^2 = 9 \quad \text{implies} \quad x = \pm\sqrt{9} = \pm 3$$

But it is incorrect to use $\pm$ when evaluating the positive square root of a number:

$$\sqrt{9} \neq \pm 3 \qquad \sqrt{9} = 3$$

EXAMPLE 1

Solving Equations Involving Radicals

Solve:

(A) $x + \sqrt{x - 4} = 4$ (B) $\sqrt{2x + 3} - \sqrt{x - 2} = 2$

SOLUTIONS

(A)

$$x + \sqrt{x - 4} = 4 \qquad \text{Isolate radical on one side.}$$

$$\sqrt{x - 4} = 4 - x \qquad \text{Square both sides.}$$

$$(\sqrt{x - 4})^2 = (4 - x)^2$$

$$x - 4 = 16 - 8x + x^2 \qquad \text{Write in standard form.}$$

$$x^2 - 9x + 20 = 0 \qquad \text{Factor left side.}$$

$$(x - 5)(x - 4) = 0 \qquad \text{Use the zero property.}$$

$$x - 5 = 0 \quad \text{or} \quad x - 4 = 0$$

$$x = 5 \quad \text{or} \quad x = 4$$

CHECK

$x = 5$	$x = 4$
$x + \sqrt{x - 4} = 4$	$x + \sqrt{x - 4} = 4$
$5 + \sqrt{5 - 4} \stackrel{?}{=} 4$	$4 + \sqrt{4 - 4} \stackrel{?}{=} 4$
$6 \neq 4$	$4 \stackrel{\checkmark}{=} 4$

This shows that 4 is a solution to the original equation and 5 is extraneous. Thus,

$$x = 4 \qquad \text{Only one solution}$$

(B) To solve an equation that contains more than one radical, isolate one radical at a time and square both sides to eliminate the isolated radical. Repeat this process until all the radicals are eliminated.

$$\sqrt{2x+3} - \sqrt{x-2} = 2 \quad \text{Isolate one of the radicals.}$$
$$\sqrt{2x+3} = \sqrt{x-2} + 2 \quad \text{Square both sides.}$$
$$(\sqrt{2x+3})^2 = (\sqrt{x-2} + 2)^2$$
$$2x + 3 = x - 2 + 4\sqrt{x-2} + 4 \quad \text{Isolate the remaining radical.}$$
$$x + 1 = 4\sqrt{x-2} \quad \text{Square both sides.}$$
$$(x+1)^2 = (4\sqrt{x-2})^2$$
$$x^2 + 2x + 1 = 16(x-2) \quad \text{Write in standard form.}$$
$$x^2 - 14x + 33 = 0 \quad \text{Factor left side.}$$
$$(x-3)(x-11) = 0 \quad \text{Use the zero property.}$$
$$x - 3 = 0 \quad \text{or} \quad x - 11 = 0$$
$$x = 3 \quad \text{or} \quad x = 11$$

CHECK

$x = 3$	$x = 11$
$\sqrt{2x+3} - \sqrt{x-2} = 2$	$\sqrt{2x+3} - \sqrt{x-2} = 2$
$\sqrt{2(3)+3} - \sqrt{3-2} \overset{?}{=} 2$	$\sqrt{2(11)+3} - \sqrt{11-2} \overset{?}{=} 2$
$2 \overset{\checkmark}{=} 2$	$2 \overset{\checkmark}{=} 2$

Both solutions check. Thus,

$$x = 3, 11 \quad \text{Two solutions}$$

MATCHED PROBLEM 1

Solve:

(A) $x - 5 = \sqrt{x-3}$

(B) $\sqrt{2x+5} + \sqrt{x+2} = 5$

› Revisiting Equations Involving Absolute Value

Squaring both sides of an equation can be a useful operation even if the equation does not involve any radicals.

EXAMPLE 2

Absolute Value Equations Revisited

Solve the following equation by squaring both sides:

$$|x+4| = 3x - 8$$

SOLUTION

$$|x + 4| = 3x - 8 \quad \text{Square both sides.}$$

$$|x + 4|^2 = (3x - 8)^2 \quad \text{Use } |u|^2 = u^2 \text{ and expand each side.}$$

$$x^2 + 8x + 16 = 9x^2 - 48x + 64 \quad \text{Write in standard form.}$$

$$8x^2 - 56x + 48 = 0 \quad \text{Divide both sides by 8.}$$

$$x^2 - 7x + 6 = 0 \quad \text{Factor the left side.}$$

$$(x - 1)(x - 6) = 0 \quad \text{Use the zero property.}$$

$$x - 1 = 0 \quad \text{or} \quad x - 6 = 0$$

$$x = 1 \quad \text{or} \quad x = 6$$

CHECK

$x = 1$	$x = 6$
$\lvert x + 4\rvert = 3x - 8$	$\lvert x + 4\rvert = 3x - 8$
$\lvert 1 + 4\rvert \overset{?}{=} 3(1) - 8$	$\lvert 6 + 4\rvert \overset{?}{=} 3(6) - 8$
$\lvert 5\rvert \overset{?}{=} -5$	$\lvert 10\rvert \overset{?}{=} 10$
$5 \neq -5$	$10 \overset{\checkmark}{=} 10$

Thus, $x = 6$ is the only solution.

Compare this solution with the solution of Example 7, Section 1-3. Squaring both sides eliminates the need to consider two separate cases.

MATCHED PROBLEM 2

Solve the following equation by squaring both sides:

$$|3x - 4| = x + 4$$

››› CAUTION ›››

When squaring an expression like $\sqrt{x - 2} + 2$, be certain to correctly apply the formula for squaring the sum of two terms (see Section R-4):

$$\begin{aligned} &(u + v)^2 = \quad u^2 \quad + \quad 2uv \quad + \quad v^2 \\ (\sqrt{x - 2} + 2)^2 &= (\sqrt{x - 2})^2 + 2(\sqrt{x - 2})(2) + (2)^2 \\ &= x - 2 + 4\sqrt{x - 2} + 4 \end{aligned}$$

Do not omit the middle term in this product:

$$(\sqrt{x - 2} + 2)^2 \neq x - 2 + 4$$

Solving Equations Involving Rational Exponents

To solve the equation

$$x^{2/3} - x^{1/3} - 6 = 0$$

write it in the form

$$(x^{1/3})^2 - x^{1/3} - 6 = 0$$

You can now recognize that the equation is quadratic in $x^{1/3}$. So, we solve for $x^{1/3}$ first, and then solve for x. We can solve the equation directly or make the substitution $u = x^{1/3}$, solve for u, and then solve for x. Both methods of solution are shown below.

Method I. Direct solution:

$$(x^{1/3})^2 - x^{1/3} - 6 = 0 \quad \text{Factor left side.}$$
$$(x^{1/3} - 3)(x^{1/3} + 2) = 0 \quad \text{Use the zero property.}$$
$$x^{1/3} = 3 \quad \text{or} \quad x^{1/3} = -2 \quad \text{Cube both sides.}$$
$$(x^{1/3})^3 = 3^3 \qquad (x^{1/3})^3 = (-2)^3$$
$$x = 27 \qquad x = -8$$

Solution set: $\{-8, 27\}$

Method II. Using substitution:
Let $u = x^{1/3}$, solve for u, and then solve for x.

$$u^2 - u - 6 = 0 \quad \text{Factor left side.}$$
$$(u - 3)(u + 2) = 0 \quad \text{Use the zero property.}$$
$$u = 3, -2$$

Replacing u with $x^{1/3}$, we obtain

$$x^{1/3} = 3 \quad \text{or} \quad x^{1/3} = -2 \quad \text{Cube both sides.}$$
$$x = 27 \qquad x = -8$$

Solution set: $\{-8, 27\}$

In general, if an equation that is not quadratic can be transformed to the form

$$au^2 + bu + c = 0$$

where u is an expression in some other variable, then the equation is said to be **quadratic in *u*** and is called an **equation of quadratic type.** Once recognized as an equation of quadratic type, an equation often can be solved using quadratic methods.

»» EXPLORE-DISCUSS 2

Which of the following is an equation of quadratic type?

(A) $3x^{-4} + 2x^{-2} + 7$ **(B)** $7x^5 - 3x^2 + 3$

(C) $2x^5 + 4x^2\sqrt{x} - 6$ **(D)** $8x^{-2}\sqrt{x} - 5x^{-1}\sqrt{x} - 2$

In general, if a, b, c, m, and n are nonzero real numbers, when is $ax^m + bx^n + c = 0$ an equation of quadratic type?

EXAMPLE 3 Solving Equations of Quadratic Type

Solve:

(A) $x^4 - 3x^2 - 4 = 0$ (B) $3x^{-2/5} - 6x^{-1/5} + 2 = 0$

SOLUTIONS

(A) The equation is quadratic in x^2. We solve for x^2 and then for x:

$$(x^2)^2 - 3x^2 - 4 = 0 \qquad \text{Factor the left side.}$$

$$(x^2 - 4)(x^2 + 1) = 0 \qquad \text{Use the zero property.}$$

$$x^2 = 4 \quad \text{or} \quad x^2 = -1 \qquad \text{Use the square root property.}$$

$$x = \pm 2 \quad \text{or} \quad x = \pm i$$

Solution set: $\{-2, 2, -i, i\}$

Since we did not raise each side of the equation to a natural number power, we do not have to check for extraneous solutions. (You should still check the accuracy of the solutions.)

(B) The equation $3x^{-2/5} - 6x^{-1/5} + 2 = 0$ is quadratic in $x^{-1/5}$. We substitute $u = x^{-1/5}$ and solve for u:

$$3u^2 - 6u + 2 = 0 \qquad \text{Use the quadratic formula.}$$

$$u = \frac{6 \pm \sqrt{12}}{6} = \frac{3 \pm \sqrt{3}}{3}$$

$$x = u^{-5} \qquad \text{Substitute } u = \frac{3 \pm \sqrt{3}}{3}.$$

$$= \left(\frac{3 \pm \sqrt{3}}{3}\right)^{-5} \qquad \text{Write with positive exponent.}$$

Thus, the two solutions are

$$x = \left(\frac{3}{3 \pm \sqrt{3}}\right)^5 \qquad \text{Two real solutions}$$

MATCHED PROBLEM 3

Solve:

(A) $x^4 + 3x^2 - 4 = 0$ (B) $3x^{-2/5} - x^{-1/5} - 2 = 0$

EXPLORE-DISCUSS 3

Solve the equation $m - 7\sqrt{m} + 12 = 0$ two ways:

(A) By squaring **(B)** By making a substitution

Which method do you prefer? Why?

EXAMPLE 4 Setting Up and Solving a Word Problem

The diagonal of a rectangle is 10 inches, and the area is 45 square inches. Find the dimensions of the rectangle correct to one decimal place.

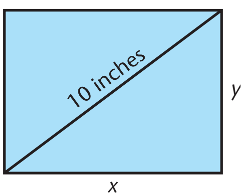


Figure 1

SOLUTION

Draw a rectangle and label the dimensions as shown in Figure 1. From the Pythagorean theorem,

$$x^2 + y^2 = 10^2$$

Thus,

$$y = \sqrt{100 - x^2}$$

Since the area of the rectangle is given by xy, we have

$$x\sqrt{100 - x^2} = 45$$ Square both sides.

$$x^2(100 - x^2) = 2{,}025$$ Remove parentheses.

$$100x^2 - x^4 = 2{,}025$$ Write as a quadratic in x^2.

$$(x^2)^2 - 100x^2 + 2{,}025 = 0$$ Use the quadratic formula.

$$x^2 = \frac{100 \pm \sqrt{100^2 - 4(1)(2{,}025)}}{2}$$ Simplify.

$$x^2 = 50 \pm 5\sqrt{19}$$ Use the square root property.

$$x = \sqrt{50 \pm 5\sqrt{19}}$$ Discard the negative solutions since $x > 0$.

If $x = \sqrt{50 + 5\sqrt{19}} \approx 8.5$, then

$$y = \sqrt{100 - x^2}$$ Substitute $x = \sqrt{50 + 5\sqrt{19}}$

$$= \sqrt{100 - (50 + 5\sqrt{19})}$$ Simplify.

$$= \sqrt{50 - 5\sqrt{19}} \approx 5.3$$

Thus, the dimensions of the rectangle to one decimal place are 8.5 inches by 5.3 inches. Notice that if $x = \sqrt{50 - 5\sqrt{19}}$, then $y = \sqrt{50 + 5\sqrt{19}}$, and the dimensions are still 8.5 inches by 5.3 inches.

CHECK Area: $(8.5)(5.3) = 45.05 \approx 45$

Diagonal: $\sqrt{8.5^2 + 5.3^2} = \sqrt{100.34} \approx 10$

Note: An exact check can be obtained by using $\sqrt{50 - 5\sqrt{19}}$ and $\sqrt{50 + 5\sqrt{19}}$ in place of these decimal approximations. This is left to the reader.

MATCHED PROBLEM 4

If the area of a right triangle is 24 square inches and the hypotenuse is 12 inches, find the lengths of the legs of the triangle correct to one decimal place.

ANSWERS TO MATCHED PROBLEMS

1. (A) $x = 7$ (B) $x = 2$
2. $x = 0, 4$
3. (A) $x = \pm 1, \pm 2i$ (B) $x = 1, -\frac{243}{32}$
4. 11.2 inches by 4.3 inches

1-6 Exercises

In Problems 1–6, determine the validity of each statement. If a statement is false, explain why.

1. If $x^2 = 5$, then $x = \pm\sqrt{5}$.
2. $\sqrt{25} = \pm 5$
3. $(\sqrt{x-1} + 1)^2 = x$
4. $(\sqrt{x} - 1)^2 + 1 = x$
5. If $x^3 = 2$, then $x = 8$.
6. If $x^{1/3} = 8$, then $x = 2$.

Solve:

7. $\sqrt{x+2} = 4$
8. $\sqrt{x-4} = 2$
9. $\sqrt{3y-2} = y - 2$
10. $\sqrt{4y+1} = 5 - y$
11. $\sqrt{5w+6} - w = 2$
12. $\sqrt{2w-3} + w = 1$
13. $|2x + 1| = x + 2$
14. $|2x + 2| = 5 - x$
15. $|x - 5| = 7 - 2x$
16. $|x + 7| = 1 - 2x$
17. $|3x - 4| = 2x - 5$
18. $|3x - 1| = x - 1$

In Problems 19–24, determine if the equation is an equation of quadratic type. Do not solve.

19. $3x^5 - 4x^2 + 9 = 0$
20. $4y^{-6} - 7y^{-3} + 17 = 0$
21. $6t^{4/5} + 11t^{2/5} - 8 = 0$
22. $5w^2 - 2w^{-1} - 4 = 0$
23. $2y + \sqrt{y} + 5 = 0$
24. $3x - \sqrt[3]{x} - 13 = 0$

In Problems 25–48, solve the equation

25. $\sqrt{3t-2} = 1 - 2\sqrt{t}$
26. $\sqrt{5t+4} - 2\sqrt{t} = 1$
27. $m^4 + 2m^2 - 15 = 0$
28. $m^4 + 4m^2 - 12 = 0$
29. $3x = \sqrt{x^2 - 2}$
30. $x = \sqrt{5x^2 + 9}$
31. $2y^{2/3} + 5y^{1/3} - 12 = 0$
32. $3y^{2/3} + 2y^{1/3} - 8 = 0$
33. $(m^2 - 2m)^2 + 2(m^2 - 2m) = 15$

34. $(m^2 + 2m)^2 - 6(m^2 + 2m) = 16$

35. $\sqrt{2t + 3} + 2 = \sqrt{t - 2}$

36. $\sqrt{2x - 1} - \sqrt{x - 5} = 3$

37. $\sqrt{w + 3} + \sqrt{2 - w} = 3$

38. $\sqrt{w + 7} = 2 + \sqrt{3 - w}$

39. $\sqrt{8 - z} = 1 + \sqrt{z + 5}$

40. $\sqrt{3z + 1} + 2 = \sqrt{z - 1}$

41. $\sqrt{4x^2 + 12x + 1} - 6x = 9$

42. $6x - \sqrt{4x^2 - 20x + 17} = 15$

43. $y^{-2} - 2y^{-1} + 3 = 0$

44. $y^{-2} - 3y^{-1} + 4 = 0$

45. $2t^{-4} - 5t^{-2} + 2 = 0$

46. $15t^{-4} - 23t^{-2} + 4 = 0$

47. $3z^{-1} - 3z^{-1/2} + 1 = 0$

48. $2z^{-1} - 3z^{-1/2} + 2 = 0$

Solve Problems 49–52 two ways: by squaring and by substitution.

49. $4m + 8\sqrt{m} - 5 = 0$

50. $4m + 8\sqrt{m} - 21 = 0$

51. $2w + 3\sqrt{w} = 14$

52. $3w + 5\sqrt{w} = 12$

In Problems 53–60, solve the equation.

53. $\sqrt{7 - 2x} - \sqrt{x + 2} = \sqrt{x + 5}$

54. $\sqrt{1 + 3x} - \sqrt{2x - 1} = \sqrt{x + 2}$

55. $3 + x^{-4} = 5x^{-2}$

56. $2 + 4x^{-4} = 7x^{-2}$

57. $2\sqrt{x + 5} = 0.01x + 2.04$

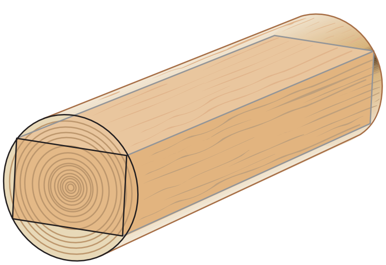
58. $3\sqrt{x - 1} = 0.05x + 2.9$

59. $2x^{-2/5} - 5x^{-1/5} + 1 = 0$

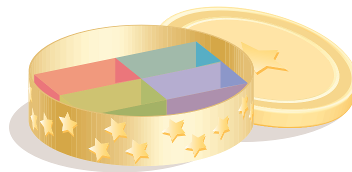
60. $x^{-2/5} - 3x^{-1/5} + 1 = 0$

APPLICATIONS

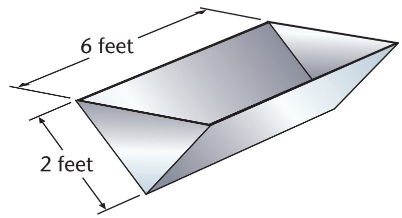
61. MANUFACTURING A lumber mill cuts rectangular beams from circular logs (see the figure). If the diameter of the log is 16 inches and the cross-sectional area of the beam is 120 square inches, find the dimensions of the cross section of the beam correct to one decimal place.



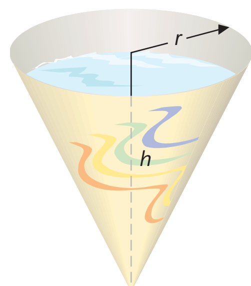
62. DESIGN A food-processing company packages an assortment of their products in circular metal tins 12 inches in diameter. Four identically sized rectangular boxes are used to divide the tin into eight compartments (see the figure). If the cross-sectional area of each box is 15 square inches, find the dimensions of the boxes correct to one decimal place.



★**63. CONSTRUCTION** A water trough is constructed by bending a 4- by 6-foot rectangular sheet of metal down the middle and attaching triangular ends (see the figure). If the volume of the trough is 9 cubic feet, find the width correct to two decimal places.



★**64. DESIGN** A paper drinking cup in the shape of a right circular cone is constructed from 125 square centimeters of paper (see the figure). If the height of the cone is 10 centimeters, find the radius correct to two decimal places.



Lateral surface area:

$S = \pi r\sqrt{r^2 + h^2}$

CHAPTER 1 Review

1-1 Linear Equations and Applications

A **solution** or **root** of an equation is a number in the **domain** or **replacement set** of the variable that when substituted for the variable makes the equation a true statement. To **solve an equation** is to find its solution set. An equation is an **identity** if it is true for all values from the domain of the variable and a **conditional equation** if it is true for some domain values and false for others. Two equations are **equivalent** if they have the same **solution set.** The **properties of equality** are used to solve equations:

1. If $a = b$, then $a + c = b + c$. **Addition Property**
2. If $a = b$, then $a - c = b - c$. **Subtraction Property**
3. If $a = b$ and $c \neq 0$, then $ca = cb$. **Multiplication Property**
4. If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$. **Division Property**
5. If $a = b$, then either may replace the other in any statement without changing the truth or falsity of statement. **Substitution Property**

› STRATEGY FOR SOLVING WORD PROBLEMS

1. Read the problem carefully—several times if necessary—that is, until you understand the problem, know what is to be found, and know what is given.
2. Let one of the unknown quantities be represented by a variable, say x, and try to represent all other unknown quantities in terms of x. This is an important step and must be done carefully.
3. If appropriate, draw figures or diagrams and label known and unknown parts.
4. Look for formulas connecting the known quantities to the unknown quantities.
5. Form an equation relating the unknown quantities to the known quantities.
6. Solve the equation and write answers to *all* questions asked in the problem.
7. Check and interpret all solutions in terms of the original problem—not just the equation found in step 5—since a mistake may have been made in setting up the equation in step 5.

An equation that can be written in the **standard form** $ax + b = 0$, $a \neq 0$, is a **linear** or **first-degree equation.**

If Q is **quantity** produced or **distance** traveled at an average or uniform **rate** R in T units of **time,** then the **quantity–rate–time formulas** are

$$R = \frac{Q}{T} \qquad Q = RT \qquad T = \frac{Q}{R}$$

1-2 Linear Inequalities

The **inequality symbols** $<, >, \leq, \geq$ are used to express **inequality relations. Line graphs, interval notation,** and the set operations of **union** and **intersection** are used to describe inequality relations. A **solution** of a linear inequality in one variable is a value of the variable that makes the inequality a true statement. Two inequalities are **equivalent** if they have the same **solution set. Inequality properties** are used to solve inequalities:

1. If $a < b$ and $b < c$, then $a < c$. **Transitive Property**
2. If $a < b$, then $a + c < b + c$. **Addition Property**
3. If $a < b$, then $a - c < b - c$. **Subtraction Property**
4. If $a < b$ and $c > 0$, then $ca < cb$.
5. If $a < b$ and $c < 0$, then $ca > cb$.
   } **Multiplication Property** (4 and 5)
6. If $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$.
7. If $a < b$ and $c < 0$, then $\frac{a}{c} > \frac{b}{c}$.
   } **Division Property** (6 and 7)

The order of an inequality reverses if we multiply or divide both sides of an inequality statement by a negative number.

1-3 Absolute Value in Equations and Inequalities

The **absolute value** of a number x is the distance on a real number line from the origin to the point with coordinate x and is given by

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

The **distance between points *A* and *B*** with coordinates a and b, respectively, is $d(A, B) = |b - a|$, which has the following **geometric interpretations:**

$\|x - c\| = d$	Distance between x and c is equal to d.
$\|x - c\| < d$	Distance between x and c is less than d.
$0 < \|x - c\| < d$	Distance between x and c is less than d, but $x \neq c$.
$\|x - c\| > d$	Distance between x and c is greater than d.

Equations and inequalities involving absolute values are solved using the following relationships for $p > 0$:

1. $|x| = p$ is equivalent to $x = p$ or $x = -p$.

2. $|x| < p$ is equivalent to $-p < x < p$.

3. $|x| > p$ is equivalent to $x < -p$ or $x > p$.

These relationships also hold if x is replaced with $ax + b$. For x any real number, $\sqrt{x^2} = |x|$.

1-4 Complex Numbers

A **complex number** in **standard form** is a number in the form $a + bi$, where a and b are real numbers and i is the **imaginary unit.** If $b \neq 0$, then $a + bi$ is also called an **imaginary number.** If $a = 0$, then $0 + bi = bi$ is also called a **pure imaginary number.** If $b = 0$, then $a + 0i = a$ is a **real number.** The **real part** of $a + bi$ is a and the **imaginary part** is bi. The complex **zero** is $0 + 0i = 0$. The **conjugate** of $a + bi$ is $a - bi$. **Equality, addition,** and **multiplication** are defined as follows:

1. $a + bi = c + di$ if and only if $a = c$ and $b = d$

2. $(a + bi) + (c + di) = (a + c) + (b + d)i$

3. $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Since complex numbers obey the same commutative, associative, and distributive properties as real numbers, most operations with complex numbers are performed by using these properties and the fact that $i^2 = -1$. The **property of conjugates,**

$$(a + bi)(a - bi) = a^2 + b^2$$

can be used to find **reciprocals** and **quotients.** If $a > 0$, then the **principal square root of the negative real number** $-a$ is $\sqrt{-a} = i\sqrt{a}$.

1-5 Quadratic Equations and Applications

A **quadratic equation** in **standard form** is an equation that can be written in the form

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

where x is a variable and a, b, and c are constants. Methods of solution include:

1. Factoring and using the **zero property:**

$$m \cdot n = 0 \quad \text{if and only if} \quad m = 0 \text{ or } n = 0 \text{ (or both)}$$

2. Using the **square root property:**

$$\text{If } A^2 = C, \text{ then } A = \pm\sqrt{C}$$

3. Completing the square:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

4. Using the **quadratic formula:**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the **discriminant $b^2 - 4ac$** is positive, the equation has two distinct **real roots;** if the discriminant is 0, the equation has one real **double root;** and if the discriminant is negative, the equation has two **imaginary roots,** each the conjugate of the other.

1-6 Additional Equation-Solving Techniques

A *square root radical* can be eliminated from an equation by isolating the radical on one side of the equation and squaring both sides of the equation. The new equation formed by squaring both sides may have **extraneous solutions.** Consequently, **every solution of the new equation must be checked in the original equation to eliminate extraneous solutions.** If an equation contains more than one radical, then the process of isolating a radical and squaring both sides can be repeated until all radicals are eliminated. If a substitution transforms an equation into the form $au^2 + bu + c = 0$, where u is an expression in some other variable, then the equation is an **equation of quadratic type** that can be solved by quadratic methods.

CHAPTER 1 Review Exercises

Work through all the problems in this chapter review and check answers in the back of the book. Answers to all review problems are there, and following each answer is a number in italics indicating the section in which that type of problem is discussed. Where weaknesses show up, review appropriate sections in the text.

Solve Problems 1 and 2.

1. $0.05x + 0.25(30 - x) = 3.3$

2. $\frac{5x}{3} - \frac{4 + x}{2} = \frac{x - 2}{4} + 1$

Solve and graph Problems 3–5.

3. $3(2 - x) - 2 \leq 2x - 1$ **4.** $|y + 9| < 5$

5. $|3 - 2x| \leq 5$

6. Find the real part, the imaginary part, and the conjugate: (A) $9 - 4i$ (B) $5i$ (C) -10

7. Perform the indicated operations and write the answers in standard form:
(A) $(-3 + 2i) + (6 - 8i)$
(B) $(3 - 3i)(2 + 3i)$
(C) $\frac{13 - i}{5 - 3i}$

Solve Problems 8–13.

8. $2x^2 - 7 = 0$ **9.** $2x^2 = 4x$

10. $2x^2 = 7x - 3$ **11.** $m^2 + m + 1 = 0$

12. $y^2 = \frac{3}{2}(y + 1)$ **13.** $\sqrt{5x - 6} - x = 0$

14. For what values of x does $\sqrt{3 - 5x}$ represent a real number?

Solve Problems 15 and 16.

15. $\frac{7}{2 - x} = \frac{10 - 4x}{x^2 + 3x - 10}$ **16.** $\frac{u - 3}{2u - 2} = \frac{1}{6} - \frac{1 - u}{3u - 3}$

Solve and graph Problems 17–19.

17. $\frac{x + 3}{8} \leq 5 - \frac{2 - x}{3}$ **18.** $|3x - 8| > 2$

19. $\sqrt{(1 - 2m)^2} \leq 3$

20. If the coordinates of A and B on a real number line are -8 and -2, respectively, find:
(A) $d(A, B)$ (B) $d(B, A)$

21. Perform the indicated operations and write the final answers in standard form:
(A) $(3 + i)^2 - 2(3 + i) + 3$ (B) i^{27}

22. Convert to $a + bi$ forms, perform the indicated operations, and write the final answers in standard form:
(A) $(2 - \sqrt{-4}) - (3 - \sqrt{-9})$
(B) $\frac{2 - \sqrt{-1}}{3 + \sqrt{-4}}$ (C) $\frac{4 + \sqrt{-25}}{\sqrt{-4}}$

Solve Problems 23–28.

23. $\left(u + \frac{5}{2}\right)^2 = \frac{5}{4}$ **24.** $1 + \frac{3}{u^2} = \frac{2}{u}$

25. $\frac{x}{x^2 - x - 6} - \frac{2}{x - 3} = 3$ **26.** $2x^{2/3} - 5x^{1/3} - 12 = 0$

27. $m^4 + 5m^2 - 36 = 0$

28. $\sqrt{y - 2} - \sqrt{5y + 1} = -3$

Solve Problems 29–33, and round answers to three significant digits.

29. $2.15x - 3.73(x - 0.930) = 6.11x$

30. $-1.52 \leq 0.770 - 2.04x \leq 5.33$

31. $|9.71 - 3.62x| > 5.48$

32. $\frac{3.77 - 8.47i}{6.82 - 7.06i}$

33. $6.09x^2 + 4.57x - 8.86 = 0$

Solve Problems 34–36 for the indicated variable in terms of the other variables.

34. $P = M - Mdt$ for M (mathematics of finance)

35. $P = EI - RI^2$ for I (electrical engineering)

36. $x = \frac{4y + 5}{2y + 1}$ for y

37. Find the error in the following "solution" and then find the correct solution.

$$\frac{4}{x^2 - 4x + 3} = \frac{3}{x^2 - 3x + 2}$$
$$4x^2 - 12x + 8 = 3x^2 - 12x + 9$$
$$x^2 = 1$$
$$x = -1 \quad \text{or} \quad x = 1$$

38. Consider the quadratic equation

$$x^2 - 6x + c = 0$$

where c is a real number. Discuss the relationship between the values of c and the three types of roots listed in Table 1 in Section 1-5.

39. For what values of a and b is the inequality $a + b < b - a$ true?

40. If a and b are negative numbers and $a > b$, then is a/b greater than 1 or less than 1?

41. Solve for x in terms of y: $y = \dfrac{1}{1 - \dfrac{1}{1 - x}}$

42. Solve and graph: $0 < |x - 6| < d$

Solve Problems 43 and 44.

43. $2x^2 = \sqrt{3}x - \frac{1}{2}$

44. $4 = 8x^{-2} - x^{-4}$

45. Evaluate: $(a + bi)\left(\dfrac{a}{a^2 + b^2} - \dfrac{b}{a^2 + b^2}i\right)$, $a, b \neq 0$

APPLICATIONS

46. **NUMBERS** Find a number such that subtracting its reciprocal from the number gives $\frac{16}{15}$.

47. **SPORTS MEDICINE** The following quotation was found in a sports medicine handout: "The idea is to raise and sustain your heart rate to 70% of its maximum safe rate for your age. One way to determine this is to subtract your age from 220 and multiply by 0.7."

(A) If H is the maximum safe sustained heart rate (in beats per minute) for a person of age A (in years), write a formula relating H and A.

(B) What is the maximum safe sustained heart rate for a 20-year-old?

(C) If the maximum safe sustained heart rate for a person is 126 beats per minute, how old is the person?

★**48.** **CHEMISTRY** A chemical storeroom has an 80% alcohol solution and a 30% alcohol solution. How many milliliters of each should be used to obtain 50 milliliters of a 60% solution?

★**49.** **RATE–TIME** An excursion boat takes 2 hours longer to go 45 miles up a river than to return. If the boat's speed in still water is 12 miles per hour, what is the rate of the current?

★**50.** **RATE–TIME** A crew of four practices by rowing up a river for a fixed distance and then returning to their starting point. The river has a current of 3 km/h.

(A) Currently the crew can row 15 km/h in still water. If it takes them 25 minutes to make the round-trip, how far upstream did they row?

(B) After some additional practice the crew cuts the round-trip time to 23 minutes. What is their still-water speed now? Round answers to one decimal place.

(C) If the crew wants to increase their still-water speed to 18 km/h, how fast must they make the round-trip? Express answer in minutes rounded to one decimal place.

51. **COST ANALYSIS** Cost equations for manufacturing companies are often quadratic in nature. If the cost equation for manufacturing inexpensive calculators is $C = x^2 - 10x + 31$, where C is the cost of manufacturing x units per week (both in thousands), find:

(A) The output for a \$15 thousand weekly cost

(B) The output for a \$6 thousand weekly cost

52. **BREAK-EVEN ANALYSIS** The manufacturing company in Problem 51 sells its calculators to wholesalers for \$3 each. Thus, its revenue equation is $R = 3x$, where R is revenue and x is the number of units sold per week (both in thousands). Find the break-even point(s) for the company—that is, the output at which revenue equals cost.

★**53.** **CHEMISTRY** If the temperature T of a solution must be kept within 5°C of 110°C, express this restriction as an absolute value inequality.

★**54.** **DESIGN** The pages of a textbook have uniform margins of 2 centimeters on all four sides (see the figure). If the area of the entire page is 480 square centimeters and the area of the printed portion is 320 square centimeters, find the dimensions of the page.

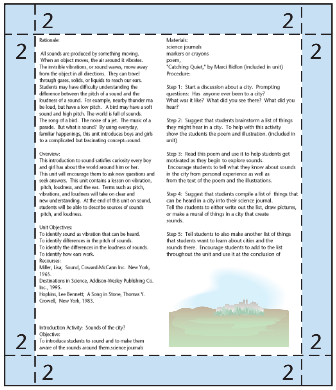
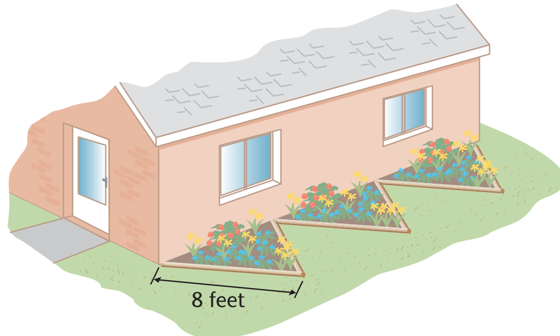


Figure for 54.

★**55.** **DESIGN** A landscape designer uses 8-foot timbers to form a pattern of isosceles triangles along the wall of a building (see the figure). If the area of each triangle is 24 square feet, find the base correct to two decimal places.



CHAPTER 1

››› GROUP ACTIVITY Solving a Cubic Equation

If a, b, and c are real numbers with $a \neq 0$, then the quadratic equation $ax^2 + bx + c = 0$ can be solved by a variety of methods, including the quadratic formula. How can we solve the cubic equation

$$ax^3 + bx^2 + cx + d = 0, \qquad a \neq 0 \tag{1}$$

and is there a formula for the roots of this equation?

The first published solution of equation (1) is generally attributed to the Italian mathematician Girolamo Cardano (1501–1576) in 1545. His work led to a complicated formula for the roots of equation (1) that involves topics that are discussed later in this text. For now, we will use Cardano's method to find a real solution in special cases of equation (1). Note that because a is nonzero, we can always multiply both sides of (1) by $1/a$ to make the coefficient of x^3 equal to 1.

To solve the cubic equation

$$x^3 - 6x^2 + 6x - 5 = 0 \tag{2}$$

first substitute

$$x = y - \frac{b}{3} = y - \frac{-6}{3} = y + 2$$

in equation (2) and simplify to obtain the *reduced cubic*

$$y^3 - 6y = 9 \tag{3}$$

Next introduce two new variables u and v defined by

$$3uv = -6$$ Note that -6 is the coefficient of y in equation (3).

and

$$u^3 - v^3 = 9$$ Note that 9 is the constant term in equation (3).

Solve the first equation for $v = -2/u$ and substitute for v in the second, to obtain

$$u^3 - \left(\frac{-2}{u}\right)^3 = 9$$ Simplify.

$$u^3 + \frac{8}{u^3} = 9$$ Multiply both sides by u^3.

$$u^6 + 8 = 9u^3$$ Subtract $-9u^3$ from both sides.

$$u^6 - 9u^3 + 8 = 0$$ Factor this quadratic in u^3.

$$(u^3 - 8)(u^3 - 1) = 0$$ Use the zero property.

$$u^3 - 8 = 0 \quad \text{or} \quad u^3 - 1 = 0$$

$$u^3 = 8 \quad \text{or} \quad u^3 = 1$$ Take the cube root of each side.

$$u = \sqrt[3]{8} = 2 \quad \text{or} \quad u = \sqrt[3]{1} = 1$$

$$v = \frac{-2}{u} = \frac{-2}{2} = -1 \quad \text{or} \quad v = \frac{-2}{u} = \frac{-2}{1} = -2$$ Use $v = -2/u$ to determine v.

If we choose either $u = 2$ and $v = -1$ or $u = 1$ and $v = -2$, then

$$y = u - v = 3$$

is a solution of equation (3) (verify this) and

$$x = y + 2 = 3 + 2 = 5$$

is a solution of equation (2) (verify this also).

The steps we followed in solving equation (2) are summarized next.

Cardano's Method for Solving a Cubic Equation

Let $x^3 + bx^2 + cx + d = 0$

Step 1. Substitute $x = y - b/3$ to obtain the *reduced cubic* $y^3 + my = n$.
Step 2. Define u and v by $m = 3uv$ and $n = u^3 - v^3$. Use $v = m/3u$ to write

$$n = u^3 - \left(\frac{m}{3u}\right)^3$$

and solve for u^3 by factoring this quadratic in u^3 or by using the quadratic formula. Then solve for u.

Step 3. Using either of the solutions found in step 2,

$$x = y - \frac{b}{3} = u - v - \frac{b}{3}$$

is a solution to $x^3 + bx^2 + cx + d = 0$

(A) The key to Cardano's method is to recognize that if u and v are defined as in step 2, then $y = u - v$ is a solution of the reduced cubic. Verify this by substituting $y = u - v$, $m = 3uv$, and $n = u^3 - y^3$ in $y^3 + my = n$ and show that the result is an identity.

(B) Use Cardano's method to solve

$$x^3 - 6x^2 - 3x - 8 = 0 \tag{4}$$

Use a calculator to find a decimal approximation of your solution and check your answer by substituting this approximate value in equation (4).

(C) Use Cardano's method to solve

$$x^3 - 6x^2 + 9x - 6 = 0 \tag{5}$$

Use a calculator to find a decimal approximation of your solution and check your answer by substituting this approximate value in equation (5).

(D) In step 2 of Cardano's method, show that u^3 is real if $\left(\frac{n}{2}\right)^2 \geq \left(\frac{-m}{3}\right)^3$.